In-Class Problems Week 11, Fri.

Problem 1.
Solve the following problems using the pigeonhole principle. For each problem, try to identify the *pigeons*, the *pigeonholes*, and a *rule* assigning each pigeon to a pigeonhole.

(a) In a certain Institute of Technology, every ID number starts with a 9. Suppose that each of the 75 students in a class sums the nine digits of their ID number. Explain why two people must arrive at the same sum.

(b) In every set of 100 integers, there exist two whose difference is a multiple of 37.

(c) For any five points inside a unit square (not on the boundary), there are two points at distance *less than* $1/\sqrt{2}$.

(d) Show that if $n + 1$ numbers are selected from $\{1, 2, 3, \ldots, 2n\}$, two must be consecutive, that is, equal to k and $k + 1$ for some k.

Problem 2.
To ensure password security, a company requires their employees to choose a password. A length 10 word containing each of the characters:

a, d, e, f, i, l, o, p, r, s,

is called a *cword*. A password can be a cword which does not contain any of the subwords "fails", "failed", or "drop."

For example, the following two words are passwords: adefiloprs, srpolifeda,
but the following three cwords are not: a**drop**eflis, **faileddrop**s, **drop**e**fails**.

(a) How many cwords contain the subword "drop"?

(b) How many cwords contain both "drop" and "fails"?

(c) Use the Inclusion-Exclusion Principle to find a simple arithmetic formula involving factorials for the number of passwords.

Problem 3.
How many paths are there from point $(0, 0)$ to $(50, 50)$ if each step along a path increments one coordinate and leaves the other unchanged? How many are there when there are impassable boulders sitting at points $(10, 11)$ and $(21, 20)$? (You do not have to calculate the number explicitly; your answer may be an expression involving binomial coefficients.)

Hint: Inclusion-Exclusion.

Supplemental problems

Problem 4. **(a)** Prove that every positive integer divides a number such as 70, 700, 7770, 77000, whose decimal representation consists of one or more 7's followed by one or more 0's.

Hint: $7, 77, 777, 7777, \ldots$

(b) Conclude that if a positive number is not divisible by 2 or 5, then it divides a number whose decimal representation is all 7's.

Problem 5.
Show that for any set of 201 positive integers less than 300, there must be two whose quotient is a power of three (with no remainder).