In-Class Problems Week 10, Mon.

Problem 1.
Recall that for functions \( f, g \) on \( \mathbb{N} \), \( f = O(g) \) iff
\[
\exists c \in \mathbb{N} \exists n_0 \in \mathbb{N} \forall n \geq n_0 \quad c \cdot g(n) \geq |f(n)|. \tag{1}
\]

For each pair of functions below, determine whether \( f = O(g) \) and whether \( g = O(f) \). In cases where one function is \( O() \) of the other, indicate the smallest nonnegative integer \( c \), and for that smallest \( c \), the smallest corresponding nonnegative integer \( n_0 \) ensuring that condition (1) applies.

(a) \( f(n) = n^2, g(n) = 3n \)
(b) \( f(n) = (3n - 7)/(n + 4), g(n) = 4 \)
(c) \( f(n) = 1 + (n \sin(n\pi/2))^2, g(n) = 3n \)

Problem 2.

(a) Indicate which of the following asymptotic relations below on the set of nonnegative real-valued functions are equivalence relations (E), strict partial orders (S), weak partial orders (W), or none of the above (N).

- \( f \sim g \), the “asymptotically equal” relation.
- \( f = o(g) \), the “little Oh” relation.
- \( f = O(g) \), the “big Oh” relation.
- \( f = \Theta(g) \), the “Theta” relation.
- \( f = O(g) \) AND NOT(\( g = O(f) \)).

(b) Indicate the implications among the assertions in part (a). For example,
\[
f = o(g) \text{ IMPLIES } f = O(g).
\]

Problem 3.

False Claim.
\[
2^n = O(1). \tag{2}
\]
Explain why the claim is false. Then identify and explain the mistake in the following bogus proof.

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Bogus proof. The proof is by induction on $n$ where the induction hypothesis, $P(n)$, is the assertion (2).

**base case:** $P(0)$ holds trivially.

**inductive step:** We may assume $P(n)$, so there is a constant $c > 0$ such that $2^n \leq c \cdot 1$. Therefore,

\[ 2^{n+1} = 2 \cdot 2^n \leq (2c) \cdot 1, \]

which implies that $2^{n+1} = O(1)$. That is, $P(n+1)$ holds, which completes the proof of the inductive step.

We conclude by induction that $2^n = O(1)$ for all $n$. That is, the exponential function is bounded by a constant.

**Supplemental problems**

**Problem 4.**
Assign true or false for each statement and prove it.

- $n^2 \sim n^2 + n$
- $3^n = O(2^n)$
- $n^{\sin(n\pi/2)+1} = o(n^2)$
- $n = \Theta\left(\frac{3n^3}{(n+1)(n-1)}\right)$

**Problem 5.**
Give an elementary proof (without appealing to Stirling’s formula) that $\log(n!) = \Theta(n \log n)$. 