Countable Sets

A is countable iff can be listed \( a_0, a_1, a_2, \ldots \)

same as \( \mathbb{N} \) bij \( A \) or \( A \) finite

so \( \mathbb{Z}^+ \), \( \mathbb{Z} \) countable

Binary words are countable

\( \{0,1\}^* \) ::= finite binary words

list the (empty) string of length 0

list the 2 length-1 bit strings

then list the \( 2^2 \) length-2 bit strings

(in binary notation order)

then the \( 2^3 \) length-3 bit strings

\( \vdots \)

\( \mathbb{N} \times \mathbb{N} \) is countable

start with \((0,0)\)

then \((0,1)\), \((1,0)\)

then \((0,2)\), \((2,0)\), \((1,1)\)

then \((0,3)\), \((3,0)\), \((1,2)\), \((2,1)\)

\( \vdots \)

then all pairs with sum \( n \)
Proving Countability

Lemma: A is countable iff can list A allowing repeats: \( \mathbb{N} \) surj A

Corollary: A is countable iff C surj A for some countable C

Rationals are countable

map \((m, n)\) to \(\frac{m}{n}\)

\(\mathbb{N} \times \mathbb{N} \) surj \(\mathbb{Q}_{\geq 0}\)

countable so countable

Reals are uncountable

But \(\{0,1\}^\omega\) and the real numbers \(\mathbb{R}\) are not countable:

next lecture.