Mathematics for Computer Science

MIT 6.042J/18.062J

Uncountable Sets

Infinite Sizes

Are all sets the same size? NO!

Cantor's Theorem

shows how to keep finding bigger infinities.

Countable Sets

A is countable iff can list it:

$a_0, a_1, a_2, \ldots.$ example:

$\{0,1\}^*$ ::= {finite bit strings}

Claim: $\{0,1\}^\omega$ ::= {∞-bit strings} is uncountable.

Diagonal Arguments

Suppose $s_0, s_1, s_2, \ldots \in \{0,1\}^\omega$

	0	1	2	3	.	.	.	n	n+1	.	.	.
s_0	0	0	1	0	.	.	.	0	0	.	.	.
s_1	0	1	1	0	.	.	.	0	1	.	.	.
s_2	1	0	0	0	.	.	.	1	0	.	.	.
s_3	1	0	1	1	.	.	.	1	1	.	.	.
.	.	.			1							
.	.		.			1						
	.			.			0					

Diagonal Arguments

Suppose $s_0, s_1, s_2, \ldots \in \{0,1\}^\omega$

	0	1	2	3	.	.	.	n	n+1	.	.	.
s_0	1	0	1	0	.	.	.	0	0	.	.	.
s_1	0	0	1	0	.	.	.	0	1	.	.	.
s_2	1	0	1	0	.	.	.	1	0	.	.	.
s_3	1	0	1	0	.	.	.	1	1	.	.	.
.	.	.			0							
.	.		.			0						
	.			.			1					

Diagonal Arguments

Suppose $s_0, s_1, s_2, \ldots \in \{0,1\}^\omega$

1 0 1 0 0 0 1 ...

...differs from every row!
So $\{0,1\}^\omega$ cannot be listed:
this diagonal sequence
will be missing

$\{0,1\}^\omega$ is uncountable

So NOT $\left(\mathbb{N} \text{ surj } \{0,1\}^\omega\right)$ and

$\{0,1\}^\omega$ surj $\mathbb{N}$ obviously

$\mathbb{N}$ "strictly smaller" than $\{0,1\}^\omega$

Strictly Smaller

A strict B ::= NOT(A surj B)

A is "strictly smaller" than B

So $\mathbb{N}$ strict $\{0,1\}^\omega$

Cantor's Theorem

A strict pow(A)

for every set, A

(finite or infinite)

Diagonal Arguments

Suppose $A = \{a, b, s, t, \ldots, d, e, \ldots\}$

$pow(A) = \{f(a), f(b), f(s), \ldots, f(d), \ldots\}$

	a	b	s	t	.	.	.	d	e	.	.	.
f(a)												.
f(b)												.
f(s)												.
f(t)												.
.												

Diagonal Arguments

Suppose $A = \{a, b, s, t, \ldots, d, e, \ldots\}$

$pow(A) = \{f(a), f(b), f(s), \ldots, f(d), \ldots\}$

	a	b	s	t	c	.	.	d	e	.	.	.
f(a)	a		s	t					e			.
f(b)	a	b			c			d				.
f(s)		b		t								.
f(t)			s	t	c			d				.
f(c)		b	s					d	e			
.												
.												

Diagonal Arguments

Suppose $A = \{a, b, s, t, \ldots, d, e, \ldots\}$

$pow(A) = \{f(a), f(b), f(s), \ldots, f(d), \ldots\}$

	a	b	s	t	c	.	.	d	e	.	.	.
f(a)	~~a~~		s	t					e			.
f(b)	a	~~b~~			c			d				.
f(s)		b	s	t								.
f(t)			s	~~t~~	c			d				.
f(c)		b	s		c			d	e			
.						.						
.							.					

Diagonal Arguments

Suppose $A = \{a, b, s, t, \ldots, d, e, \ldots\}$

$pow(A) = \{f(a), f(b), f(s), \ldots, f(d), \ldots\}$

D	a	b	s	t	c	.	.	d	e	.	.	.
f(a)			s	t					e			.
f(b)	a				c			d				.
f(s)		b	s	t								.
f(t)			s		c			d				.
f(c)		b	s		c			d	e			
.						.						
.							.					

A strict Pow(A)

Pf: say have fcn $f: A \to pow(A)$.
Define a subset of A that is not in the range of f: namely

$$D ::= \{a \in A \mid a \notin f(a)\}$$

Now $D \notin range(f)$ since it differs from set $f(a)$ at element a!

A strict Pow(A)

So no f-arrow into D.

f is not a surjection.

QED

$\mathbb{N}$ strict pow($\mathbb{N}$)

That is,

pow($\mathbb{N}$) is uncountable

Proving Uncountability

Lemma: If A is uncountable and C surj A then C is uncountable

$\{0,1\}^\omega$ again

We know

$$\{0,1\}^\omega \text{ bij } pow(\mathbb{N})$$

and $pow(\mathbb{N})$ uncountable by Cantor,

$\{0,1\}^\omega$ again

We know

$$\{0,1\}^\omega \text{ bij } pow(\mathbb{N})$$

and $pow(\mathbb{N})$ uncountable by Cantor, so $\{0,1\}^\omega$ uncountable.

Real Numbers Uncountable

$$\mathbb{R} \text{ surj } \{0,1\}^\omega$$

map $\pm r$ to binary rep

7 1/3 = 111.010101...

maps to 111010101...

Real Numbers Uncountable

$$\mathbb{R} \text{ surj } \{0,1\}^{\omega}$$

map $\pm r$ to binary rep

1/2 = .100000...

1/2 maps to 100000...

= .0111111...

-1/2 maps to 0111111...