

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Mathematics for Computer Science

MIT 6.042J/18.062J

Uncountable Sets



Albert R Meyer, March 4, 2015

Cantor.1

6	9	13	7
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Infinite Sizes

Are all sets the same size? **NO!**

Cantor's Theorem

shows how to keep finding bigger infinities.



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Cantor.2

6	9	13	7
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Countable Sets

A is countable iff can list it:

a_0, a_1, a_2, \dots example:

$\{0,1\}^*$::= {finite bit strings}

Claim: $\{0,1\}^\omega$::= { ∞ -bit strings} is **uncountable**.



Albert R Meyer, March 4, 2015

Cantor.3

6	9	13	7
12		10	5
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Diagonal Arguments

Suppose $s_0, s_1, s_2, \dots \in \{0,1\}^\omega$

	0	1	2	3	...	n	n+1	...
s_0	0	0	1	0	...	0	0	...
s_1	0	1	1	0	...	0	1	...
s_2	1	0	0	0	...	1	0	...
s_3	1	0	1	1	...	1	1	...
	.	.	.	1
	1
	0



Albert R Meyer, March 4, 2013

Cantor.4

6	9	13	7
12		10	5
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Diagonal Arguments

Suppose $s_0, s_1, s_2, \dots \in \{0,1\}^\omega$

	0	1	2	3	...	n	n+1	...
s_0	1	0	1	0	...	0	0	...
s_1	0	0	1	0	...	0	1	...
s_2	1	0	1	0	...	1	0	...
s_3	1	0	1	0	...	1	1	...
...				0				
...					0			
...						1		



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Cantor.5

6	9	13	7
12		10	5
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Diagonal Arguments

Suppose $s_0, s_1, s_2, \dots \in \{0,1\}^\omega$

...differs from every row!
 So $\{0,1\}^\omega$ cannot be listed:
 this diagonal sequence
 will be missing



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Cantor.6

6	9	13	7
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$\{0,1\}^\omega$ is uncountable

So NOT $\left(\mathbb{N} \text{ surj } \{0,1\}^\omega \right)$ and
 $\{0,1\}^\omega \text{ surj } \mathbb{N}$ obviously

\mathbb{N} "strictly smaller" than $\{0,1\}^\omega$



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Cantor.8

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Strictly Smaller

A strict B ::= NOT(A surj B)
 A is "strictly smaller" than B

So \mathbb{N} strict $\{0,1\}^\omega$



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Cantor.9

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Cantor's Theorem

A strict $\text{pow}(A)$
for every set, A
(finite or infinite)



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Cantor.10

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Diagonal Arguments

Suppose $A = \{a, b, s, t, \dots, d, e, \dots\}$
 $\text{pow}(A) = \{f(a), f(b), f(s), \dots, f(d), \dots\}$

	a	b	s	t	.	.	d	e	.	.
f(a)										.
f(b)										.
f(s)										.
f(t)										.
.										.
.										.



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Cantor.11

6	9	13	7
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Diagonal Arguments

Suppose $A = \{a, b, s, t, \dots, d, e, \dots\}$
 $\text{pow}(A) = \{f(a), f(b), f(s), \dots, f(d), \dots\}$

	a	b	s	t	c	.	.	d	e	.	.
f(a)	a		s	t					e		.
f(b)	a	b			c			d			.
f(s)		b		t							.
f(t)			s	t	c			d			.
f(c)		b	s					d	e		.
.											.
.											.



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Cantor.12

6	9	13	7
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Diagonal Arguments

Suppose $A = \{a, b, s, t, \dots, d, e, \dots\}$
 $\text{pow}(A) = \{f(a), f(b), f(s), \dots, f(d), \dots\}$

	a	b	s	t	c	.	.	d	e	.	.
f(a)	a		s	t					e		.
f(b)	a	b			c			d			.
f(s)		b	s	t							.
f(t)			s	t	c			d			.
f(c)		b	s		c			d	e		.
.											.
.											.



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Cantor.13

6	9	13	7
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Diagonal Arguments

Suppose $A = \{a, b, s, t, \dots, d, e, \dots\}$
 $\text{pow}(A) = \{f(a), f(b), f(s), \dots, f(d), \dots\}$

	a	b	s	t	c	.	.	d	e	.	.
D											
f(a)			s	t				e		.	.
f(b)	a				c			d		.	.
f(s)		b	s	t						.	.
f(t)			s		c			d		.	.
f(c)		b	s		c			d	e	.	.
.										.	.
.										.	.



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Cantor.14

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A strict Pow(A)

Pf: say we have fcn $f: A \rightarrow \text{pow}(A)$.
 Define a subset of A that is not in
 the range of f : namely

$$D ::= \{a \in A \mid a \notin f(a)\}$$

Now $D \notin \text{range}(f)$ since it differs
 from set $f(a)$ at element a !



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Cantor.15

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A strict Pow(A)

So no f -arrow into D .
 f is not a surjection.
QED



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Cantor.21

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\mathbb{N} strict pow(\mathbb{N})

That is,
 $\text{pow}(\mathbb{N})$ is uncountable



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Cantor.22

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Proving **U**ncountability

Lemma: If A is **u**ncountable
and $C \text{ surj } A$ then
 C is **u**ncountable



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Cantor.24

6	9	13	7
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$\{0,1\}^\omega$ again

We know

$\{0,1\}^\omega$ bij $\text{pow}(\mathbb{N})$
and $\text{pow}(\mathbb{N})$ uncountable by
Cantor,



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Cantor.26

6	9	13	7
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3	1	4	14
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$\{0,1\}^\omega$ again

We know

$\{0,1\}^\omega$ bij $\text{pow}(\mathbb{N})$
and $\text{pow}(\mathbb{N})$ uncountable by
Cantor, so $\{0,1\}^\omega$ uncountable.



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Cantor.27

6	9	13	7
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Real Numbers Uncountable

$\mathbb{R} \text{ surj } \{0,1\}^\omega$

map $\pm r$ to binary rep
 $7 \frac{1}{3} = 111.010101\dots$
maps to $111010101\dots$



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Cantor.28

6	9	13	7
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Real Numbers Uncountable

$$\mathbb{R} \text{ surj } \{0,1\}^{\omega}$$

map $\pm r$ to binary rep

$$1/2 = .100000\dots$$

$1/2$ maps to $100000\dots$

$$= .011111\dots$$

$-1/2$ maps to $011111\dots$

