Set Theory: ZFC

Zermelo-Frankel Set Theory
Axioms of Zermelo-Frankel with the Choice axiom (ZFC) define the standard Theory of Sets

Some Axioms of Set Theory
Extensionality
\( x \) and \( y \) have the same elements

\[ \forall x [x \in y \iff x \in z] \]
iff
\( x \) and \( y \) are members of the same sets
Some Axioms of Set Theory

Extensionality

\[ \forall x \left[ x \in y \iff x \in z \right] \]

iff

\[ \forall x \left[ y \in x \iff z \in x \right] \]

Power set

Every set has a power set

\[ \forall x \exists p \forall s. s \subseteq x \iff s \in p \]

Comprehension

If \( S \) is a set, and \( P(x) \) is a predicate of set theory, then

\[ \{ x \in s \mid P(x) \} \]

is a set

Sets are Well Founded

According to ZF, the elements of a set have to be “simpler” than the set itself. In particular,

no set is a member of itself,

or a member of a member…
Sets are Well Founded

Def: $x$ is $\in$-minimal in $y$

$x$ is in $y$, but no element of $x$ is in $y$

Some Axioms of Set Theory

Foundation

Every nonempty set has an $\in$-minimal element

Sets are Well Founded

Def: $x$ is $\in$-minimal in $y$

$x \in y$ AND

$[\forall z. z \in x \implies z \notin y]$
Let $R := \{S\}$. If $S \in S$, then $R$ has no $\in$-minimal element. If it exists, it must be $S$, but $S \in R$ and $S \in S$, so $S$ is not $\in$-minimal in $R$.

Zermelo-Frankel Set Theory

$S \not\in S$ implies that

1. the collection of all sets is not a set, and
2. $W = \{s \in \text{Sets} \mid s \not\in s\}$ is the collection of all sets -- which is why it's not a set.