

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Mathematics for Computer Science

MIT 6.042J/18.062J

Set Theory: Russell Paradox



Albert R Meyer, March 4, 2015

russell.1

6	9	13	7
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Self application

Self application is notoriously doubtful:

"This statement is false."
is it **true** or **false**?



Albert R Meyer, March 4, 2015

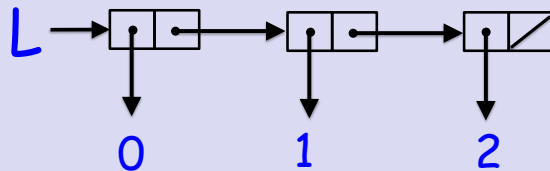
russell.2

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Self membership

The list

$L = (0\ 1\ 2)$



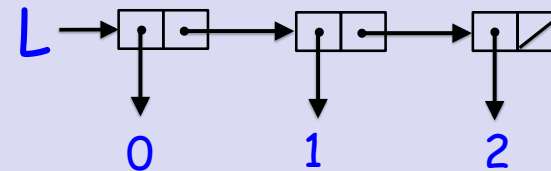
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russell.5

6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Self membership

(setcar (second L) L)



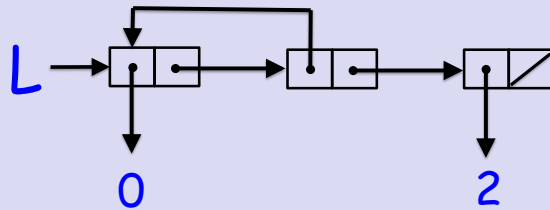
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russell.6

6	9	13	7
12		10	5
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15	8	11	2

Self membership

(setcar (second L) L)



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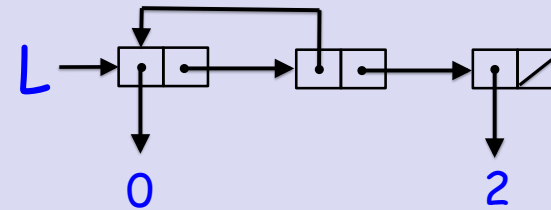
russell.7

6	9	13	7
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Self membership

Lists are member of themselves:

$L = (0 L 2)$



Albert R Meyer, March 4, 2015

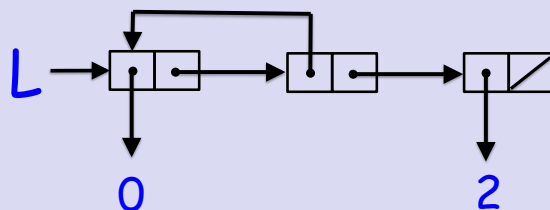
russell.8

6	9	13	7
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Self membership

Lists are member of themselves:

$L = (0 (0 (0 \dots 2) 2) 2)$



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russell.9

6	9	13	7
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Self application

compose procedures

(define (compose f g)

(define (h x)

(f (g x)))

h)



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russell.11

6	9	13	7
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Self application

compose procedures

$((\text{compose square add1}) 3)$

$\Rightarrow 16 \quad (= (3 + 1)^2)$

$((\text{compose square square}) 3)$

$\Rightarrow 81 \quad (= (3^2)^2)$



Albert R Meyer, March 4, 2015

russell.12

6	9	13	7
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Self application

compose procedures

$(\text{define (comp2 f)}$

$\text{ (compose f f)})$

$((\text{comp2 square}) 3)$

$\Rightarrow 81$



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russell.13

6	9	13	7
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Self application

apply procedure to itself:

$((\text{comp2 comp2}) \text{add1}) 3)$

$\Rightarrow 7$

$((\text{comp2 comp2}) \text{square}) 3)$

$\Rightarrow 43046721 \quad (= 3^{16})$



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russell.14

6	9	13	7
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Russell's Paradox

Let $W ::= \{s \in \text{Sets} \mid s \notin s\}$

so $[s \in W \text{ IFF } s \notin s]$

Now let s be W , and reach a contradiction:

$[W \in W \text{ IFF } W \notin W]$



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russell.15

6	9	13	7
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Disaster: Math is broken!

I am the Pope,
Pigs fly,
and verified programs
crash...



Albert R Meyer, March 4, 2015

russell.16

6	9	13	7
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...but paradox is buggy

Assumes that W is a set!

$$[s \in W \text{ IFF } s \notin s]$$

for all sets s

...can only substitute
 W for s if W is a set



Albert R Meyer, March 4, 2015

russell.17

6	9	13	7
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...but paradox is buggy

Assumes that W is a set!

We can avoid the paradox,
if we deny that W is a set!
...which raises the key question:
just which well-defined
collections are sets?



Albert R Meyer, March 4, 2015

russell.18

6	9	13	7
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Zermelo-Frankel Set Theory

No simple answer, but the
axioms of Zermelo-Frankel
along with the Choice axiom
(ZFC) do a pretty good job.



Albert R Meyer, March 4, 2015

russell.19