More (Un)Countable Sets

The Real Numbers are Uncountable

Decimal expansions:

\[ \sqrt{2} = 1.4142\ldots, \quad 5 = 5.000\ldots \]
\[ \frac{1}{10} = 0.1000\ldots, \quad \frac{1}{3} = 0.333\ldots \]
\[ \frac{1}{99} = 0.010101\ldots \]

Proving Uncountability

Lemma.

If \( U \) is an uncountable set and \( A \) surj \( U \),
then \( A \) is uncountable.

The Real Numbers are Uncountable

\( b(r) ::= 0.1 \) decimals of \( r \)

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The Real Numbers are Uncountable

\[ b : \mathbb{R} \rightarrow \{0,1\}^\omega \]

is a surjective function

So \( \mathbb{R} \) is uncountable

Proving countability

Lemma.
If \( C \) is a countable set and \( C \text{ surj } A \),
then \( A \) is countable.

Sequences of positive ints
Sequences of $\mathbb{Z}^+$

e(n) ::= \text{exponents of primes in the factorization of } n

e(3^4 \cdot 7^{22} \cdot 23^{11}) = (4, 22, 11)

\[ N \text{ surj } (\mathbb{Z}^+)^* \]

So $(\mathbb{Z}^+)^*$ is countable