Staff Solutions to Problem Set 7

Reading: Matchings, Coloring, Trees, 11.5, 11.7 through 11.10 (omit Stable Marriage, 11.6); Sums and Products, 13.1 through 13.5 (omit Double Sums, 13.6).

Problem 1.
Scholars through the ages have identified twenty fundamental human virtues: honesty, generosity, loyalty, prudence, completing the weekly course reading-response, etc. At the beginning of the term, every student in Math for Computer Science possessed exactly eight of these virtues. Furthermore, every student was unique; that is, no two students possessed exactly the same set of virtues. The Math for Computer Science course staff must select one additional virtue to impart to each student by the end of the term. Prove that there is a way to select an additional virtue for each student so that every student is unique at the end of the term as well.

Suggestion: Use Hall’s theorem. Try various interpretations for the vertices on the left and right sides of your bipartite graph.

Solution. Construct a bipartite graph $G$ as follows. The vertices on the left are all students and the vertices on the right are all subsets of nine virtues. There is an edge between a student and a set of 9 virtues if the student already has 8 of those virtues.

Each vertex on the left has degree 12, since each student can learn one of 12 additional virtues. The vertices on the right have degree at most 9, since each set of 9 virtues has only 9 subsets of size 8. So this bipartite graph is degree-constrained, and therefore, by Lemma 11.5.6, there is a matching for the students. Thus, if each student is taught the additional virtue in the set of 9 virtues with whom he or she is matched, then every student is unique at the end of the term.

Problem 2.
Let $D = (d_1, d_2, \ldots, d_n)$ be a sequence of positive integers where $n \geq 2$.

(a) Suppose $D$ is a list of the degrees of vertices of some $n$-vertex tree $T$, that is, $d_i$ is the degree of the $i$th vertex of $T$. Explain why

$$\sum_{i=1}^{n} d_i = 2(n - 1) \quad (1)$$

Solution. By the Handshaking Lemma, the sum of the degrees of the vertices of $T$ is twice the number of edges, and since $T$ is a tree with $n$ vertices, it has $n - 1$ edges.

(b) Prove conversely that if $D$ satisfies equation (1), then $D$ is a list of the degrees of the vertices of some $n$-vertex tree. Hint: Induction.

Solution. The proof will be by induction on $n$ with induction hypothesis

$$P(n) ::= \forall D. \text{ D satisfies (1)} \text{ IMPLIES } \exists \text{ tree } T. \text{ D is a list of the vertex degrees of } T.$$
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Figure 1  Coloring using 4 colors

Base case \((n = 2)\): In this case \(D\) can only be the length two list \((1, 1)\), which is the degree sequence of a 2-vertex tree.

Induction step: Suppose \(D = (d_1, d_2, \ldots, d_n)\) satisfies (1). First notice that there must be an \(i\) such that \(d_i = 1\), otherwise we would have that \(d_i \geq 2\) for all \(i\) and the sum would be too large. Similarly, there must be a \(j\) such that \(d_j \geq 2\), otherwise the sum would be too small. Now we can assume wlog that \(d_n = 1\) and \(d_{n-1} \geq 2\).

Let \(k := n - 1\) and \(D' = (d_1, d_2, \ldots, d_{k-1}, d_k - 1)\). Observe that the sum of \(D'\) equals \(2(n - 1) - 1 - 1 = 2(k - 1)\), so \(D'\) satisfies equation (1) for the case that \(n\) is \(k\). Assuming the induction hypothesis, \(P(k)\), we know that there is a \(k\)-vertex tree \(T'\) with degree sequence \(D'\). Let \(T\) be the \(n\)-vertex tree obtained by adding a new leaf to \(T'\) adjacent to the \(k\)th vertex of \(T'\). Then \(D\) is a list of the vertex degrees of \(T\), concluding the proof.

(c) Assume that \(D\) satisfies equation (1). Show that it is possible to partition \(D\) into two sets \(S_1, S_2\) such that the sum of the elements in each set is the same. Hint: Trees are bipartite.

Solution. Using the previous part we know that there is a tree \(T\) with degree sequence \(D\). As trees are bipartite graphs, there exists a partition of the vertices into sets \(V_1, V_2\) such that any edge connects a vertex in \(V_1\) with a vertex in \(V_2\). We argue that the sum of the degrees of the vertices in \(V_1\) is equal to the number of edges of the graph. The reason is that any edge in the graph contributes to 1 in the degree in exactly one of the vertices in \(V_1\). Similarly, the sum of the degrees of the vertices in \(V_2\) is also the number of edges. Thus the degrees corresponding to the partition \(V_1, V_2\) determine the sets \(S_1, S_2\) we were looking for.

A proof by induction similar to part (c) is also possible.

Problem 3.

A basic example of a simple graph with chromatic number \(n\) is the complete graph on \(n\) vertices, that is \(\chi(K_n) = n\). This implies that any graph with \(K_n\) as a subgraph must have chromatic number at least \(n\). It’s a common misconception to think that, conversely, graphs with high chromatic number must contain a large complete subgraph. In this problem we exhibit a simple example countering this misconception, namely a graph with chromatic number four that contains no triangle—length three cycle—and hence no subgraph isomorphic to \(K_n\) for \(n \geq 3\). Namely, let \(G\) be the 11-vertex graph of Figure 2. The reader can verify that \(G\) is triangle-free.

(a) Show that \(G\) is 4-colorable.

Solution. Figure 1 shows a valid coloring.
Figure 2  Graph $G$ with no triangles and $\chi(G) = 4$.

Figure 3  Constraints on a 3-coloring.

(b) Prove that $G$ can’t be colored with 3 colors.

Solution. Assume by contradiction that there is one coloring using only 3 colors: red, blue and green. The outer pentagon of the graph is an odd-length cycle, and so requires all 3 colors. So we can assume wlog that the outer pentagon is colored as shown in the left hand side of Figure 3.

This coloring of the pentagon forces the coloring of three interior points, as shown in the right hand side of Figure 3. Now the point in the center has neighbors with all three colors, so it is impossible to color it.

Problem 4.
Use integration to find upper and lower bounds that differ by at most 0.1 for the following sum. (You may need to add the first few terms explicitly and then use integrals to bound the sum of the remaining terms.)

$$\sum_{i=1}^{\infty} \frac{1}{(2i + 1)^2}$$

Solution. Let’s first try standard bounds:

$$\int_0^\infty \frac{1}{(2x + 3)^2} \, dx \leq \sum_{i=1}^{\infty} \frac{1}{(2i + 1)^2} \leq \int_0^\infty \frac{1}{(2x + 1)^2} \, dx$$

Evaluating the integrals gives:

$$-\frac{1}{2(2x + 3)} \bigg|_0^\infty \leq \sum_{i=1}^{\infty} \frac{1}{(2i + 1)^2} \leq -\frac{1}{2(2x + 1)} \bigg|_0^\infty$$
\[
\frac{1}{6} \leq \sum_{i=1}^{\infty} \frac{1}{(2i+1)^2} \leq \frac{1}{2}
\]

These bounds are too far apart, so let’s sum the first couple terms explicitly and bound the rest with integrals.

\[
\frac{1}{3^2} + \frac{1}{5^2} + \int_{2}^{\infty} \frac{1}{(2x + 3)^2} \, dx \leq \sum_{i=1}^{\infty} \frac{1}{(2i + 1)^2} \leq \frac{1}{3^2} + \frac{1}{5^2} + \int_{2}^{\infty} \frac{1}{(2x + 1)^2} \, dx
\]

Integration now gives:

\[
\frac{1}{3^2} + \frac{1}{5^2} + \left( -\frac{1}{2(2x + 3)} \right|_{2}^{\infty} \right) \leq \sum_{i=1}^{\infty} \frac{1}{(2i + 1)^2} \leq \frac{1}{3^2} + \frac{1}{5^2} + \left( -\frac{1}{2(2x + 1)} \right|_{2}^{\infty}
\]

\[
\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{14} \leq \sum_{i=1}^{\infty} \frac{1}{(2i + 1)^2} \leq \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{10}
\]

Now we have bounds that differ by \(1/10 - 1/14 < 1/10 = 0.1\).