Staff Solutions to Miniquiz 4-28

Problem 1 (Counting) (1 point).
Let \( f_n \) denote the \( n \)th non-negative even integer, i.e., \( f_0 = 0, f_1 = 2, f_2 = 4 \), and so on.

(a) Give a linear recurrence for \( f_n \).

Solution.
\[
f_n = f_{n-1} + 2. \tag{1}
\]

(b) Derive a closed form for the generating function \( F(x) \) for the sequence \( f_0, f_1, \ldots \), i.e,
\[
F(x) := f_0 + f_1 x + f_2 x^2 + \ldots + f_n x^n + \ldots.
\]

Solution. We have:
\[
F(x) = f_0 + f_1 x + \ldots + f_n x^n + \ldots \quad \text{and} \quad -xF(x) = -f_0 x - f_1 x^2 - \ldots - f_{n-1} x^n + \ldots.
\]

Therefore, we get:
\[
F(x)(1-x) = f_0 + (f_1 - f_0)x + (f_2 - f_1)x^2 + \ldots
\]
\[
= 0 + 2x + 2x^2 + \ldots, \quad \text{from the recurrence (1)}
\]
\[
= \frac{2x}{1-x}.
\]

So,
\[
F(x) = \frac{2x}{(1-x)^2}.
\]

Problem 2 (Discrete Probability) (1 point).
The Yankees and the Red Sox are playing a two-out-of-three series; in other words, they play until one team has won two games. Assume that the Red Sox win each game with probability \( \frac{3}{5} \), regardless of the outcomes of previous games.

Answer the questions below using the four-step method. Exhibit the tree diagram, assign probabilities to each outcome, and calculate the required probabilities. Use the same tree diagram for both problems.

(a) What is the probability that the series goes three games?
Solution. From the tree diagram, we get:

\[ \Pr[3 \text{ games played}] = \frac{12}{125} + \frac{18}{125} + \frac{12}{125} + \frac{18}{125} = \frac{12}{25}. \]

Alternatively: consider the event when the series ends in 2. Then, either the Yankees win both games, which occurs with probability \((2/5)^2\) or the Red Sox win both games, which occurs with probability \((3/5)^2\). Summing these yields \((4/25) + (9/25) = 13/25\). Therefore, the probability that the series goes 3 games = 
\[ 1 - \frac{13}{25} = \frac{12}{25}. \]

(b) What is the probability that the Red Sox win the series?

Solution. From the tree diagram, we get:

\[ \Pr[\text{Sox Win}] = \frac{18}{125} + \frac{18}{125} + \frac{9}{25} = \frac{81}{125}. \]