Staff Solutions to Miniquiz 4-14

Problem 1 (Practice with Big-Oh) (1 point).

Recall that if \( f \) and \( g \) are nonnegative real-valued functions on \( \mathbb{Z}^+ \), then \( f = O(g) \) iff there exist \( c, n_0 \in \mathbb{Z}^+ \) such that

\[
\forall n \geq n_0. \ f(n) \leq c g(n).
\]

For each pair of functions \( f \) and \( g \) below, indicate the smallest \( c \in \mathbb{Z}^+ \), and for that smallest \( c \), the smallest corresponding \( n_0 \in \mathbb{Z}^+ \), that would establish \( f = O(g) \) by the definition given above. If there is no such \( c \), write \( \infty \).

(a) \( f(n) = \frac{1}{2} \ln n^2, \ g(n) = n \).

Solution. \( f(n) = \ln n, \) and \( n \) exceeds \( \ln n \) for all positive \( n \). Thus \( c = 1 \) and \( n_0 = 1 \).

(b) \( f(n) = n, \ g(n) = n \ln n \).

Solution. Since \( \ln n \) eventually grows beyond 1, it must be that \( n \ln n \) eventually grows beyond \( n \). Thus \( c = 1 \). Now \( f(n) = n \leq c g(n) = g(n) = n \ln n \) precisely when \( 1 \leq \ln n \). That is, when \( n \geq e \). So \( n_0 = \lfloor e \rfloor = 3 \).

(c) \( f(n) = 2^n, \ g(n) = n^4 \ln n \).

Solution. \( n^4 \ln n = o\left(n^5\right) \) since \( \ln n = o(n) \). Also, any polynomial is asymptotically smaller than any exponential whose base has magnitude greater than 1. So \( n^5 = o(2^n) \) and hence \( n^4 \ln n = o(2^n) \). Therefore \( f \neq O(g) \), so there do not exist finite \( c, n_0 \in \mathbb{Z}^+ \) that satisfy the required condition. Thus, here we write \( c = \infty \).

(d) \( f(n) = 3 \sin \left( \frac{\pi (n - 1)}{100} \right) + 2, \ g(n) = 0.2 \).

Solution. \( f(n) \) is periodic. Its minimum value is \( -1 \) and its maximum is \( 5 \), so the smallest acceptable positive integer value for \( c \) is \( \frac{5}{0.2} = 25 \). Now, \( c g(n) \) exceeds or equals \( f(n) \) for all positive \( n \), so \( n_0 = 1 \).
Problem 2 (Bijections) (1 point).
Suppose $n$ books are lined up on a shelf. The number of selections of $m$ of the books so that selected books are separated by at least three unselected books is the same as the number of all length $k$ binary strings with exactly $m$ ones.

(a) What is the value of $k$?

Solution.

$$k = n - 3(m - 1).$$

(b) Describe a bijection between between the set of all length $k$ binary strings with exactly $m$ ones and such book selections.

Solution. A selection of $m$ among $n$ books on a shelf corresponds in an obvious way to a length $n$ binary string with exactly $m$ ones. So we need a bijection from length $k$ strings with $m$ ones to length $n$ strings with $m$ one’s that are at least three apart. Such a bijection is defined by replacing each of the first $m - 1$ ones in the length $k$ string by $1000$. 

\[ \blacksquare \]