Staff Solutions to Mini-Quiz 2-24

Problem 1.
You’ve seen how certain set identities follow from corresponding propositional equivalences. For example, you proved by a chain of iff’s that

\[(A - B) \cup (A \cap B) = A\]

using the fact that the propositional formula \((P \land \overline{Q}) \lor (P \land Q)\) is equivalent to \(P\).

State a similar propositional equivalence that would justify the key step in a proof for the following set equality organized as a chain of iff’s:

\[(A \cup B) - (A - B) = B\]  \hspace{1cm} (1)

(You are not being asked to write out an iff-proof of the equality or to write out a proof of the propositional equivalence. Just state the equivalence.)

Solution. The needed propositional equivalence is that

\[(P \lor Q) \land \overline{P} \land Q\] is equivalent to \(Q\).

This problem illustrates the clear correspondence set equalities involving operations, like union and set difference, and corresponding propositional equivalences. The correspondence reduces set equality proofs to proofs of propositional equivalence, allowing for automatic proofs of such set equalities.

Problem 2. (a) Five assertions about a binary relation \(R : A \rightarrow B\) are bulleted below. There are six predicate formulas that express some of these assertions. Write the number of each formula next to the bulleted assertion it expresses, if there is one. For example, you should write “2” next to the last bulleted assertion, since formula 2 expresses that \(R\) is the identity relation. Write “none” next to an assertion that no formula expresses.

Variables \(a, a_1, \ldots\) range over the domain \(A\) and \(b, b_1, \ldots\) range over the codomain \(B\). More than one formula may express the same bulleted assertion, and some formulas may not express any bulleted assertion.

- \(R\) is a surjection
  
  Solution. 3

- \(R\) is an injection
  
  Solution. 5

- \(R\) is a function
  
  Solution. 4
• \( R \) is total

Solution. 1

• \( R \) is the identity relation.

Solution. 2

1. \( \forall a. \exists b. a R b. \)
2. \( \forall a, b. a R b \iff a = b. \)
3. \( \forall b. \exists a. a R b. \)
4. \( \forall a_1, a_2, b_1, b_2. (a_1 R b_1 \text{ AND } a_2 R b_2 \text{ AND } b_1 \neq b_2) \text{ IMPLIES } a_1 \neq a_2. \)
5. \( \forall a_1, a_2, b. (a_1 R b \text{ AND } a_2 R b) \text{ IMPLIES } a_1 = a_2. \)

(b) Give an example of a relation \( R \) that satisfies three of the properties — surjection, injection, total, and function (you indicate which) — but is not a bijection.

Solution. Let

\[
A ::= \{1, 2\}, \; B ::= \{1\}, \; \text{graph}(R) ::= \{(1, 1)\}.
\]

Then \( R \) is not a bijection because it is not total, and indeed \(|A| \neq |B|\). But \( R \) is an injective, surjective function.

There can be several other possible answers.