Staff Solutions to Midterm Exam March 3

Problem 1 (Proof by Contradiction) (20 points).
Prove that $\log_{12} 18$ is irrational.

Solution. Proof. Suppose to the contrary that

$$\log_{12} 18 = \frac{m}{n}$$

for some integers $m, n$ where $n > 0$. So we have

$$12^{\log_{12} 18} = 12^{m/n} \quad \text{(raising 12 to equal powers)},$$

$$18 = 12^{m/n} \quad \text{(def of log)},$$

$$18^n = 12^m \quad \text{(raising both sides to the nth power)},$$

$$(2 \cdot 3^2)^n = (2^2 \cdot 3)^m \quad \text{(factoring 18 & 12 into primes)}.$$  \quad (1)

Now we have two cases:

Case 1: $(n \neq 2m)$. There are different numbers of two’s on the left and right hand sides of equation (1), which contradicts the Unique Factorization Theorem.

Case 2: $(n = 2m)$. There are $2n$ three’s on the left hand side of (1) and $n/2$ three’s on the right hand side, which contradicts the Unique Factorization Theorem.

In any case there is a contradiction, which implies that $\log_{12} 18$ must be irrational.

An alternative to the argument by cases is to assume (for the sake of contradiction) that $n = 2m$. Then $\log_{12} 18 = 1/2$, which is false (since $12^{1/2} \neq 18$). This implies that Case 1 is in fact the only possibility.

Problem 2 (Logical Formulas) (20 points).
Suppose $P(n)$ is a predicate on the nonnegative integers such that

$$\forall n. P(n) \text{ IMPLIES } P(n + 2).$$  \quad (2)

For each of the assertions (a)–(d) below, determine whether

- the assertion Can be true for some $P$ and false for others,
- the assertion Always is true for any $P$, or
- the assertion Never is true for any $P$. 

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Indicate which case applies by circling the correct letter. For assertions that can be both true and false, describe a predicate $P(n)$ satisfying (2) for which the expression is false.

(a) $P(1)$ IMPLIES $\forall n. P(2n + 1)$

Solution. A. This assertion says that if $P(1)$ holds, then $P(n)$ holds for all odd $n$. This case is always true.

(b) $\neg(P(0))$ AND $\forall n \geq 1. P(n)$

Solution. C. Let $P(n)$ always be true. Then (2) holds trivially, but this assertion is false since $P(0)$ is true.

(c) $\exists n. \exists m > n. [P(2n) \text{ AND } \neg P(2m)]$

Solution. N. This assertion says that $P$ holds for some even number, $2n$, but not for some other larger even number, $2m$. However, if $P(2n)$ holds, we can apply (2) $n-m$ times to conclude $P(2m)$ also holds. This case is impossible.

(d) $\exists n. \exists m < n. [P(2n) \text{ AND } \neg P(2m)]$

Solution. C. If $P(n)$ is true for all $n$, then (2) holds, but this assertion is false because $P(2m)$ is never false. The assertion can also be true, for example, if $P(2m)$ is false for $m = 0$ but true everywhere else.

Problem 3 (Induction) (20 points).

You are given $n$ envelopes, numbered $0, 1, \ldots, n - 1$. Envelope 0 contains $2^0 = 1$ dollar, Envelope 1 contains $2^1 = 2$ dollars, $\ldots$, and Envelope $n-1$ contains $2^{n-1}$ dollars. Let $P(n)$ be the assertion that:

For all nonnegative integers $k < 2^n$, there is a subset of the $n$ envelopes whose contents total to exactly $k$ dollars.

Prove by induction that $P(n)$ holds for all integers $n \geq 1$.

Solution. Base case ($n = 1$): The only possible values of $k$ are 0 and 1. The sole envelope contains the needed $\$1$ and the empty subset—that is, not using any envelopes—gives $\$0$.

Inductive step Assume that $P(n)$ is true for some $n \geq 1$. We need to show that, given $n+1$ envelopes, there is a subset that sums to exactly $k$ dollars for $0 \leq k < 2^{n+1}$.

There are two cases to consider:

1. $k < 2^n$. By the Induction Hypothesis, we can get exactly $k$ dollars using a subset of the first $k$ envelopes.

2. $2^n \leq k < 2^{n+1}$. Then, $k = j + 2^n$, where $0 \leq j < 2^n$. Now by Induction Hypothesis, we can get exactly $j$ dollars using a subset of the first $n$ envelopes. Then this subset along with the $(n+1)$th envelope sums to $j + 2^n = k$.

In either case, we are able to get exactly $k$ dollars using a subset of the envelopes. Therefore, $P(n+1)$ is true. By induction, we conclude that $P(n)$ is true for all $n \geq 1$.

NOTE: In the following question, the defs of $S$ & $T$ in part (c) were reversed in the actual exam, which trivialized the question, since no transition is possible from the state in which all numbers are in order.
Problem 4 (State Machines) (20 points).

The following problem is a twist on the Fifteen-Puzzle problem that we did in class.

Let $A$ be a sequence consisting of the numbers $1, \ldots, n$ in some order. A pair of integers in $A$ is called an out-of-order pair when the first element of the pair both comes earlier in the sequence, and is larger, than the second element of the pair. For example, the sequence $(1, 2, 4, 5, 3)$ has two out-of-order pairs: $(4, 3)$ and $(5, 3)$. We let $t(A)$ equal the number of out-of-order pairs in $A$. For example, $t((1, 2, 4, 5, 3)) = 2$.

The elements in $A$ can be rearranged using the Rotate-Triple operation, in which three consecutive elements of $A$ are rotated to move the smallest of them to be first.

For example, in the sequence $(2, 4, 1, 5, 3)$, the Rotate-Triple operation could rotate the consecutive numbers $4, 1, 5$, into $1, 5, 4$ so that

$$(2, 4, 1, 5, 3) \rightarrow (2, 1, 5, 4, 3).$$

The Rotate-Triple could also rotate the consecutive numbers $2, 4, 1$ into $1, 2, 4$ so that

$$(2, 4, 1, 5, 3) \rightarrow (1, 2, 4, 5, 3).$$

We can think of a sequence $A$ as a state of a state machine whose transitions correspond to possible applications of the Rotate-Triple operation.

(a) Argue that the derived variable $t$ is weakly decreasing.

Solution. Suppose the Rotate-Triple operation is applied to three consecutive elements $a, b, c$ in $A$. This has no effect on the out-of-order pairs involving at most one of $a, b$ and $c$.

To analyze pairs where both elements are one of $a, b$ and $c$, there are two cases.

If $b$ is the smallest element, then $a, b, c$ get rearranged into $b, c, a$. This has the effect of reversing the two pairs $(a, b), (a, c)$ into $(b, a), (c, a)$. If $a < c$, this causes a net change of zero in $t(A)$, while if $c < a$, this causes a net decrease of two in $t(A)$.

If $c$ is the smallest, then $a, b, c$ get rearranged into $c, a, b$. This has the effect of reversing the two pairs $(a, c), (b, c)$ into $(c, a), (c, b)$, which similarly leaves $t(A)$ unchanged or decreased by two.

So in each case, $t$ is either constant or decreases, showing that $t$ is weakly decreasing.

(b) Prove that having an even number of out-of-order pairs is a preserved invariant of this machine.

Solution. This part follows directly from the argument in the previous part showing that $t$ changes by $0$ or $-2$. So if the number of out-of-order pairs is even, then it stays even.

(c) Starting with

$$S ::= (2014, 2013, 2012, \ldots, 2, 1),$$

explain why it is impossible to reach

$$T ::= (1, 2, \ldots, 2012, 2013, 2014).$$

Solution. Since $T$ has no out-of-order pairs, $t(T) = 0$. On the other hand, all the pairs in $S$ are out-of-order, so $t(S) = (2014 \cdot 2013)/2$ which is odd. Since parity is preserved, $T$ cannot be reachable from $S$. 
Problem 5 (Relations) (10 points).
Let $R : A \to B$ and $S : B \to C$ be binary relations such that $S \circ R$ is a bijection and $|A| = 2$.

Give an example of such $R$, $S$ where neither $R$ nor $S$ is a function.

*Hint:* Let $|B| = 4$.

**Solution.** It is easy to see that $R$ must be total ($\geq 1$ out) and $S$ must be a surjection ($\geq 1$ in).

$C$ must have 2 elements since $A$ bij $C$.

For minimal example with $R$, $S$ not functions, see Figure 1. For an example in which $R$ is not even an injection, see Figure 2.

To explain: following the hint, we let $|B| = 4$.

We put an $R$-arrow from the first element of $A$ to the second element of $B$ and an $S$-arrow from this second element of $B$ to the first element of $C$; likewise from the second element of $A$ to the third element of $B$ and from this third element of $B$ to the second element of $C$. These arrows will provide the bijection defined by $S \circ R$.

We put two $R$-arrows into the first element in $B$ with no $S$-arrow out as in Figure 2; so $R$ is not a function or an injection and $S$ is not total. We put two $S$-arrows out of the fourth element of $B$ and no $R$-arrows in;
so $R$ is not a surjection and $S$ is not a function or an injection.

So besides $R$ being total and $S$ a surjection, Figure 2 shows that neither $R$ nor $S$ need have any additional “jection” properties.

Problem 6 (Cardinality) (10 points).

For each of the following sets, indicate whether it is finite (F), countably infinite (C), or uncountable (U).

1. The set of solutions to the equation $x^3 - x = -0.1$.
2. The set of natural numbers $\mathbb{N}$.
3. The set of rational numbers $\mathbb{Q}$.
4. The set of real numbers $\mathbb{R}$.
5. The set of integers $\mathbb{Z}$.
6. The set of complex numbers $\mathbb{C}$.
7. The set of words in the English language no more than 20 characters long.
8. The powerset of the set of all possible bijections from $\{1, 2, \ldots, 10\}$ to itself.
9. An infinite set $S$ with the property that there exists a total surjective function $f : \mathbb{N} \to S$.
10. A set $A \cup B$ where $A$ is countable and $B$ is uncountable.

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