Staff Solutions to In-Class Problems Week 8, Fri.

STAFF NOTE: Asymptotics, Ch. 13.7 (13.6 skipped)

Problem 1.
Recall that for functions \( f, g \) on \( \mathbb{N} \), \( f = O(g) \) iff
\[
\exists c \in \mathbb{N} \forall n_0 \in \mathbb{N} \forall n \geq n_0 \quad c \cdot g(n) \geq |f(n)|.
\] (1)

For each pair of functions below, determine whether \( f = O(g) \) and whether \( g = O(f) \). In cases where one function is \( O() \) of the other, indicate the smallest nonnegative integer, \( c \), and for that smallest \( c \), the smallest corresponding nonnegative integer \( n_0 \) ensuring that condition (1) applies.

(a) \( f(n) = n^2, g(n) = 3n \).

Solution. ((\( f = O(g) \):) NO, because \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty \).

Solution. ((\( g = O(f) \):) YES, with \( c = 1, n_0 = 3 \), which works because \( 3^2 = 9, 3 \cdot 3 = 9 \).

(b) \( f(n) = (3n - 7)/(n + 4), g(n) = 4 \)

Solution. ((\( f = O(g) \):) YES, with \( c = 1, n_0 = 0 \) (because \( |f(n)| < 3 \)).

Solution. ((\( g = O(f) \):) YES, with \( c = 2, n_0 = 15 \).

Since \( \lim_{n \to \infty} f(n) = 3 \), the smallest possible \( c \) is 2. For \( c = 2 \), the smallest possible \( n_0 = 15 \) which follows from the requirement that \( 2f(n_0) \geq 4 \).

(c) \( f(n) = 1 + (n \sin(n\pi/2))^2, g(n) = 3n \)

Solution. ((\( f = O(g) \):) NO, because \( f(2n + 1) = (2n + 1)^2 + 1 \neq O(n) \) which rules out \( f = O(g) \).

Solution. ((\( g = O(f) \):) NO, because \( f(2n) = 1 \), which rules out \( g = O(f) \) since \( g = \Theta(n) \).

Problem 2.

(a) Indicate which of the following asymptotic relations below on the set of nonnegative real-valued functions are equivalence relations, (E), strict partial orders (S), weak partial orders (W), or none of the above (N).

- \( f \sim g \), the “asymptotically equal” relation.
Solution. E

- \( f = o(g) \), the “little Oh” relation.

Solution. S

- \( f = O(g) \), the “big Oh” relation.

Solution. N because it is neither symmetric nor antisymmetric.

- \( f = \Theta(g) \), the “Theta” relation.

Solution. E

- \( f = O(g) \) AND \( \text{NOT}(g = O(f)) \).

Solution. S.

(b) Indicate the implications among the assertions in part (a). For example,

\[ f = o(g) \text{ IMPLIES } f = O(g). \]

Solution.

\[ f \sim g \text{ IMPLIES } f = \Theta(g) \text{ IMPLIES } f = O(g), \]
\[ f = o(g) \text{ IMPLIES } f = O(g) \text{ AND } \text{NOT}(g = O(f)). \]

(e) Define two functions \( f, g \) that are incomparable under big Oh:

\[ f \neq O(g) \text{ AND } g \neq O(f). \]

Solution. One example is,

\[ f(n) := \begin{cases} n & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases} \quad g(n) := \begin{cases} 0 & \text{if } n \text{ is odd,} \\ n & \text{if } n \text{ is even,} \end{cases} \]

which can also be described by the formulas

\[ f(n) := n \sin \left( \frac{n\pi}{2} \right), \quad g(n) := n \cos \left( \frac{n\pi}{2} \right). \]

Problem 3.

False Claim.

\[ 2^n = O(1). \]  \hspace{1cm} (2)

Explain why the claim is false. Then identify and explain the mistake in the following bogus proof.
Bogus proof. The proof is by induction on \( n \) where the induction hypothesis, \( P(n) \), is the assertion (2).

**base case:** \( P(0) \) holds trivially.

**inductive step:** We may assume \( P(n) \), so there is a constant \( c > 0 \) such that \( 2^n \leq c \cdot 1 \). Therefore,

\[
2^{n+1} = 2 \cdot 2^n \leq (2c) \cdot 1,
\]

which implies that \( 2^{n+1} = O(1) \). That is, \( P(n + 1) \) holds, which completes the proof of the inductive step.

We conclude by induction that \( 2^n = O(1) \) for all \( n \). That is, the exponential function is bounded by a constant.

\[\square\]

**Solution.** A function is \( O(1) \) iff it is bounded by a constant, and since the function \( 2^n \) grows unboundedly with \( n \), it is not \( O(1) \).

The mistake in the bogus proof is in its misinterpretation of the expression \( 2^n \) in assertion (2). The intended interpretation of (2) is

Let \( f \) be the function defined by the rule \( f(n) ::= 2^n \). Then \( f = O(1) \). \( (3) \)

But the bogus proof treats (2) as an assertion, \( P(n) \), about \( n \). Namely, it misinterprets (2) as meaning:

Let \( f_n \) be the constant function equal to \( 2^n \). That is, \( f_n(k) ::= 2^n \) for all \( k \in \mathbb{N} \). Then

\[ f_n = O(1). \] (4)

Now (4) is true since every constant function is \( O(1) \), and the bogus proof is an unnecessarily complicated, but correct, proof that that for each \( n \), the constant function \( f_n \) is \( O(1) \). But in the last line, the bogus proof switches from the misinterpretation (4) and claims to have proved (3).

So you could say that the exact place where the proof goes wrong is in its first line, where it defines \( P(n) \) based on misinterpretation (4) and claims to have proved (3).

Supplemental problem

**Problem 4.**

Assign true or false for each statement and prove it.

- \( n^2 \sim n^2 + n \)
- \( 3^n = O\left(2^n\right) \)
- \( n^{\sin(n\pi/2)+1} = o(n^2) \)
- \( n = \Theta\left(\frac{3n^3}{(n+1)(n-1)}\right) \)

**Solution.** The 1st and 4th statements are true.

- \( \frac{n^2+n}{n^2} = \frac{n^2}{n^2} + \frac{n}{n^2} = 1 + \frac{1}{n} \), so as \( n \) approaches infinity, the ratio approaches \( 1 + 0 = 1 \). Therefore the two expressions are asymptotically equal.
- \( \lim_{n \rightarrow \infty} \frac{2^n}{3^n} = \lim_{n \rightarrow \infty} (2/3)^n = 0 \), so \( 2^n = o(3^n) \) and therefore \( \text{NOT}(3^n = O(2^n)) \).
• The left side never exceeds \( n^2 \), but when \( n = 1, 5, 9, 13, \ldots \), the left side is equal to \( n^2 \), and so is not \( o(n^2) \).

\[
\lim_{n \to \infty} \frac{n}{\frac{3n^3}{(n+1)(n-1)}} = \lim_{n \to \infty} \frac{n(n + 1)(n - 1)}{3n^3} = \frac{1}{3},
\]

so \( n = O\left(\frac{3n^3}{(n+1)(n-1)}\right) \). Similarly,

\[
\lim_{n \to \infty} \frac{\frac{3n^3}{(n+1)(n-1)}}{n} = 3,
\]

so \( \frac{3n^3}{(n+1)(n-1)} = O(n) \). Because the two expressions are big-O of each other, \( n = \Theta\left(\frac{3n^3}{(n+1)(n-1)}\right) \).