Staff Solutions to In-Class Problems Week 7, Mon.

STAFF NOTE: Acyclic Digraphs, Scheduling, Partial Orders & Equivalence Relations Ch.9.5-9.10 (omit 9.7)

Problem 1.
The table below lists some prerequisite information for some subjects in the MIT Computer Science program (in 2006). This defines an indirect prerequisite relation that is a DAG with these subjects as vertices.

\[
\begin{align*}
18.01 & \rightarrow 6.042 & 18.01 & \rightarrow 18.02 \\
18.01 & \rightarrow 18.03 & 6.046 & \rightarrow 6.840 \\
8.01 & \rightarrow 8.02 & 6.001 & \rightarrow 6.034 \\
6.042 & \rightarrow 6.046 & 18.03, 8.02 & \rightarrow 6.002 \\
6.001, 6.002 & \rightarrow 6.003 & 6.001, 6.002 & \rightarrow 6.004 \\
6.004 & \rightarrow 6.033 & 6.033 & \rightarrow 6.857
\end{align*}
\]

(a) Explain why exactly six terms are required to finish all these subjects, if you can take as many subjects as you want per term. Using a greedy subject selection strategy, you should take as many subjects as possible each term. Exhibit your complete class schedule each term using a greedy strategy.

Solution. It helps to have a diagram of the direct prerequisite relation:

\[
\begin{align*}
18.01 & \rightarrow 6.042 & 18.01 & \rightarrow 18.02 \\
18.01 & \rightarrow 18.03 & 6.046 & \rightarrow 6.840 \\
8.01 & \rightarrow 8.02 & 6.001 & \rightarrow 6.034 \\
6.042 & \rightarrow 6.046 & 18.03, 8.02 & \rightarrow 6.002 \\
6.001, 6.002 & \rightarrow 6.003 & 6.001, 6.002 & \rightarrow 6.004 \\
6.004 & \rightarrow 6.033 & 6.033 & \rightarrow 6.857
\end{align*}
\]

There is a chain of length six: \(8.01, 8.02, 6.002, 6.004, 6.033, 6.857\)

So six terms are necessary, because at most one of these subjects can be taken each term.

There is no longer chain, so with the greedy strategy you will take six terms. Here are the subjects you take in successive terms.

\[\]

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(b) In the second term of the greedy schedule, you took five subjects including 18.03. Identify a set of five subjects not including 18.03 such that it would be possible to take them in any one term (using some nongreedy schedule). Can you figure out how many such sets there are?

Solution. We're looking for an antichain that does not include 18.03. Every such antichain will have to include 18.02, 6.003, 6.034. Then a fourth subject could be any of 6.042, 6.046, and 6.840. The fifth subject could then be any of 6.004, 6.033, and 6.857. This gives a total of nine antichains of five subjects.

(e) Exhibit a schedule for taking all the courses—but only one per term.

Solution. We're asking for a topological sort of the vertices. There are many. One is 18.01, 8.01, 6.001, 18.02, 6.042, 18.03, 8.02, 6.034, 6.046, 6.002, 6.840, 6.004, 6.003, 6.033, 6.857.

(d) Suppose that you want to take all of the subjects, but can handle only two per term. Exactly how many terms are required to graduate? Explain why.

Solution. There are \([15/2] = 8\) terms necessary. The schedule below shows that 8 terms are sufficient as well:

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<td>4</td>
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<td>8</td>
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(e) What if you could take three subjects per term?

Solution. From part (a) we know six terms are required even if there is no limit on the number of subjects per term. Six terms are also sufficient, as the following schedule shows:

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Problem 2.
A pair of Math for Computer Science Teaching Assistants, Lisa and Annie, have decided to devote some of their spare time this term to establishing dominion over the entire galaxy. Recognizing this as an ambitious project, they worked out the following table of tasks on the back of Annie’s copy of the lecture notes.

1. **Devise a logo** and cool imperial theme music - 8 days.
2. **Build a fleet** of Hyperwarp Stardestroyers out of eating paraphernalia swiped from Lobdell - 18 days.
3. **Seize control** of the United Nations - 9 days, after task #1.
4. **Get shots** for Lisa’s cat, Tailspin - 11 days, after task #1.
5. **Open a Starbucks chain** for the army to get their caffeine - 10 days, after task #3.
6. **Train an army** of elite interstellar warriors by dragging people to see *The Phantom Menace* dozens of times - 4 days, after tasks #3, #4, and #5.
7. **Launch the fleet** of Stardestroyers, crush all sentient alien species, and establish a Galactic Empire - 6 days, after tasks #2 and #6.
8. **Defeat Microsoft** - 8 days, after tasks #2 and #6.

We picture this information in Figure 1 below by drawing a point for each task, and labelling it with the name and weight of the task. An edge between two points indicates that the task for the higher point must be completed before beginning the task for the lower one.

![Graph representing the task precedence constraints.](image)

(a) Give some valid order in which the tasks might be completed.
Solution. We can easily find several of them. The most natural one is valid, too: #1, #2, #3, #4, #5, #6, #7, #8.

Lisa and Annie want to complete all these tasks in the shortest possible time. However, they have agreed on some constraining work rules.

- Only one person can be assigned to a particular task; they cannot work together on a single task.
- Once a person is assigned to a task, that person must work exclusively on the assignment until it is completed. So, for example, Lisa cannot work on building a fleet for a few days, run to get shots for Tailspin, and then return to building the fleet.

(b) Lisa and Annie want to know how long conquering the galaxy will take. Annie suggests dividing the total number of days of work by the number of workers, which is two. What lower bound on the time to conquer the galaxy does this give, and why might the actual time required be greater?

Solution.

\[
\frac{8 + 18 + 9 + 11 + 10 + 4 + 6 + 8}{2} = 37 \text{ days}
\]

If working together and interrupting work on a task were permitted, then this answer would be correct. However, the rules may prevent Lisa and Annie from both working all the time. For example, suppose the only task was building the fleet. It will take 18 days, not 18/2 days, to complete, because only one person can work on it and the other must sit idle.

(c) Lisa proposes a different method for determining the duration of their project. She suggests looking at the duration of the critical path, the most time-consuming sequence of tasks such that each depends on the one before. What lower bound does this give, and why might it also be too low?

Solution. The longest sequence of tasks is devising a logo (8 days), seizing the U.N. (9 days), opening a Starbucks (10 days), training the army (4 days), and then defeating Microsoft (8 days). Since these tasks must be done sequentially, galactic conquest will require at least 39 days.

If there were enough workers, this answer would be correct; however, with only two workers, Lisa and Annie may be unable to make progress on the critical path every day. For example, suppose there were only four tasks: devise logo, build fleet, seize control, get shots. Now the critical path consists of two critical tasks: devise logo, get shots, which take 19 days. But to get through this path in 19 days, some worker must be working on a critical task at all times for the 19 days. This leaves only one worker free to complete building the fleet and seizing control, which will take at least 27 days. So in fact, 27 days is the minimum time for two workers to complete these four tasks.

(d) What is the minimum number of days that Lisa and Annie need to conquer the galaxy? No proof is required.

Solution. 40 days. Tasks could be divided as follows:

Annie: #1 (days 1-8), #3 (days 9-17), #4 (days 18-28), #8 (days 33-40).
Lisa: #2 (days 1-18), #5 (days 19-28), #6 (days 29-32), #7 (days 33-38).

It takes some care to verify that 40 days is the best you can do. If someone comes up with a simple proof of this, tell the course staff.
Problem 3.
For each of the binary relations below, state whether it is a strict partial order, a weak partial order, an equivalence relation or none of these. If it is a partial order, state whether it is a linear order. If it is none, indicate which of the axioms for partial order and equivalence relations it violates.

STAFF NOTE: This problem took longer than expected for students to go through in the class. Parts (f) and (h) are the trickiest parts, usually where students made mistakes. Give hints as needed to get them through faster.

(a) The superset relation, $\supseteq$ on the power set $\mathcal{P}\{1, 2, 3, 4, 5\}$.

Solution. This is a weak partial order, but not a linear one. For example, the sets of size 3 form an antichain.

(b) The relation between any two nonnegative integers, $a, b$ that $a \equiv b \pmod{8}$.

Solution. An equivalence relation.

(c) The relation between propositional formulas, $G, H$, that $[G \implies H]$ is valid.

Solution. Violates antisymmetry: $P$ and $\neg(\neg P)$ imply each other but are different expressions. It is transitive, though.

(d) The relation between propositional formulas, $G, H$, that $[G \iff H]$ is valid.

Solution. An equivalence relation.


Solution. Obviously violates transitivity. Asymmetric, and hence also irreflexive and antisymmetric.

(f) The empty relation on the set of real numbers.

Solution. It’s vacuously asymmetric and transitive, so it’s a strict partial order. It’s not linear. It not an equivalence relation because it is not reflexive.

(g) The identity relation on the set of integers.

Solution. It’s obviously reflexive, antisymmetric and transitive, so it’s a weak partial order. It’s not linear. It’s also an equivalence relation since it is symmetric as well.

(h) The divisibility relation on the integers, $\mathbb{Z}$.

Solution. Not antisymmetric: 3 and -3 divide each other. It is transitive and reflexive.
Problem 4.
Let $S$ be a sequence of $n$ different numbers. A subsequence of $S$ is a sequence that can be obtained by deleting elements of $S$.

For example, if

$$S = (6, 4, 7, 9, 1, 2, 5, 3, 8)$$

Then 647 and 7253 are both subsequences of $S$ (for readability, we have dropped the parentheses and commas in sequences, so 647 abbreviates $(6, 4, 7)$, for example).

An increasing subsequence of $S$ is a subsequence of whose successive elements get larger. For example, 1238 is an increasing subsequence of $S$. Decreasing subsequences are defined similarly; 641 is a decreasing subsequence of $S$.

(a) List all the maximum length increasing subsequences of $S$, and all the maximum length decreasing subsequences.

Solution. The maximum length increasing subsequences are 1238 and 1258. The maximum length decreasing subsequences are

$$641, 642, 643, 653, 753, 953$$

Now let $A$ be the set of numbers in $S$. (So $A$ is the integers $[1, 9]$ for the example above.) There are two straightforward linear orders for $A$. The first is numerical order where $A$ is ordered by the $<$ relation. The second is to order the elements by which comes first in $S$; call this order $<_S$. So for the example above, we would have

$$6 <_S 4 <_S 7 <_S 9 <_S 1 <_S 2 <_S 5 <_S 3 <_S 8$$

Let $<$ be the product relation of the linear orders $<_S$ and $<$. That is, $<$ is defined by the rule

$$a < a' \iff a < a' \text{ AND } a <_S a'.$$

So $<$ is a partial order on $A$ (Section 9.9).

(b) Draw a diagram of the partial order, $<$, on $A$. What are the maximal and minimal elements?

Solution. The maximal elements are 8 and 9; the minimal are 1, 4, and 6:

(c) Explain the connection between increasing and decreasing subsequences of $S$, and chains and antichains under $<$.

Solution. A chain, with its elements listed in numerically increasing order, is an increasing subsequence and an antichain, with its elements listed in numerically decreasing order, is a decreasing subsequence.

(d) Prove that every sequence, $S$, of length $n$ has an increasing subsequence of length greater than $\sqrt{n}$ or a decreasing subsequence of length at least $\sqrt{n}$.

Solution. By Dilworth's Lemma, either a chain or an antichain must have size at least $\sqrt{n}$, which, by the previous problem part, means there is either an increasing or a decreasing subsequence of this size.