Staff Solutions to In-Class Problems Week 12, Mon.

STAFF NOTE: Conditional Probability, Ch. 17

In Spring ’13, most teams finished 20 or more minutes early, mainly because staff was not adequately briefed to get discussions going as directed in individual problems. Also, PS_01:conditional_probability_problem_errors did not make it into the actual handout used in class.

Problem 1.
There is a rare and incurable disease called Beaver Fever, which afflicts about 1 in 1000 people. Those with this disease unrelentingly tell math jokes in social settings, believing other people would think they’re funny. Doctor Meyer has some fairly reliable tests for this disease:

- If a person has Beaver Fever, the probability that Meyer diagnoses the person as having the disease is 0.99.
- If a person doesn’t have it, the probability that Meyer diagnoses that person as not having Beaver Fever is 0.97.

Let $B$ be the event that a randomly chosen person has Beaver Fever, and $Y$ be the event that Meyer’s diagnosis is “Yes, that person has Beaver Fever,” with $\overline{B}$ and $\overline{Y}$ being the complements of these events.

(a) The description above explicitly gives the values of the following quantities. What are their values?

\[
\begin{align*}
\Pr[B] & = 0.001 \\
\Pr[Y | B] & = 0.99 \\
\Pr[\overline{Y} | \overline{B}] & = 0.97
\end{align*}
\]

Solution. \( \Pr[B] = 0.001 \), \( \Pr[Y | B] = 0.99 \), \( \Pr[\overline{Y} | \overline{B}] = 0.97 \)

(b) Write formulas for \( \Pr[\overline{B}] \) and \( \Pr[Y | \overline{B}] \) solely in terms of the explicitly given quantities in part (a) - literally use their expressions, not their numeric values.

Solution. \( \Pr[\overline{B}] = 1 - \Pr[B] \), \( \Pr[Y | \overline{B}] = 1 - \Pr[\overline{Y} | \overline{B}] \).

(c) Write a formula for the probability that Doctor Meyer says a person has the disease solely in terms of \( \Pr[B] \), \( \Pr[\overline{B}] \), \( \Pr[Y | B] \) and \( \Pr[Y | \overline{B}] \).

Solution. By the Total Probability Law:

\[
\Pr[Y] = \Pr[Y | B] \Pr[B] + \Pr[Y | \overline{B}] \Pr[\overline{B}]
\]

The values turn out to be \( 0.99(1/1000) + 0.03(1 - 1/1000) = 0.03096 \).
(d) Write a formula solely in terms of the expressions given in part (a) for the probability that a person has Beaver Fever given that Doctor Meyer says the person has it.

Solution.

$$
\Pr[B \mid Y] = \frac{\Pr[B \text{ and } Y]}{\Pr[Y]}
$$

$$
= \frac{\Pr[Y \mid B] \Pr[B]}{\Pr[Y \mid B] \Pr[B] + \Pr[Y \mid \overline{B}] \Pr[\overline{B}]}
$$

$$
= \frac{\Pr[Y \mid B] \Pr[B]}{\Pr[Y \mid B] \Pr[B] + (1 - \Pr[Y \mid B])(1 - \Pr[B])}.
$$

The values turn out to be

$$
\Pr[B \mid Y] = \frac{0.99(1/1000)}{0.03096} = \frac{99}{3096} \approx \frac{1}{32}.
$$

The low probability of actually having Beaver Fever even though the (97% accurate) test says you do is because there are way more people without the disease than those with the disease. Among 1000 people, the number of false positives (999 × 3%) is more than 30 times the number of true positives (1 × 99%). So if the test says you have Beaver Fever, it’s probably a false positive.

Of course Doctor Meyer has a recourse to a 99.9% accurate test that has no false positives: simply telling everyone they don’t have the disease.

Problem 2.

There are three prisoners in a maximum-security prison for fictional villains: the Evil Wizard Voldemort, the Dark Lord Sauron, and Little Bunny Foo-Foo. The parole board has declared that it will release two of the three, chosen uniformly at random, but has not yet released their names. Naturally, Sauron figures that he will be released to his home in Mordor, where the shadows lie, with probability $2/3$.

A guard offers to tell Sauron the name of one of the other prisoners who will be released (either Voldemort or Foo-Foo). If the guard has a choice of naming either Voldemort or Foo-Foo (because both are to be released), he names one of the two with equal probability.

Sauron knows the guard to be a truthful fellow. However, Sauron declines this offer. He reasons that knowing what the guards says will reduce his chances, so he is better off not knowing. For example, if the guard says, “Little Bunny Foo-Foo will be released”, then his own probability of release will drop to $1/2$ because he will then know that either he or Voldemort will also be released, and these two events are equally likely.

Dark Lord Sauron has made a typical mistake when reasoning about conditional probability. Using a tree diagram and the four-step method, explain his mistake. What is the probability that Sauron is released given that the guard says Foo-Foo is released?

**Hint:** Define the events $S$, $F$, and “$F$” as follows:

- “$F$” = Guard says Foo-Foo is released
- $F$ = Foo-Foo is released
- $S$ = Sauron is released

**Solution.** Sauron’s mistake can be explained as his confusing the two different events $F$ and “$F$”. His observation that $\Pr[S \mid F] = 1/2$ is correct, but that’s the wrong thing to calculate. He should be calculating $\Pr[S \mid “F”]$. 

To clarify the error and work out the proper probability, let’s begin by working out the sample space, noting events of interest, and computing outcome probabilities:

\[
\begin{array}{c|c|c|c|c}
 & F & 1/3 & \times & \times \\
F, S & F & 1/6 & \times & \\
 & V & 1/2 & & \\
F, V & 1/6 & & & \\
 & V & 1/6 & \times & \\
V, S & V & & & \\
 released & 1/3 & & & \\

guard says prob. & guard says & Foo-foo released & Sauron released
\end{array}
\]

The outcomes in each of these events are noted in the tree diagram.

The tree shows how the event \( F \) (Foo-foo will be released) is different from the event “\( F \)” (the guard says Foo-foo will be released). In particular, the probability that Sauron is released, given that Foo-foo is released, is indeed \( 1/2 \):

\[
\Pr[S \mid F] = \frac{\Pr[S \cap F]}{\Pr[F]} = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{6}} = \frac{1}{2}
\]

But the probability that Sauron is released given that the guard actually says so is still \( 2/3 \):

\[
\Pr[S \mid "F"] = \frac{\Pr[S \cap "F"]}{\Pr["F"]} = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{6}} = \frac{2}{3}
\]

So Sauron’s probability of release is unchanged by the guard’s statement.

**Problem 3.**

There are two decks of cards. One is complete, but the other is missing the Ace of spades. Suppose you pick one of the two decks with equal probability and then select a card from that deck uniformly at random. What is the probability that you picked the complete deck, given that you selected the eight of hearts? Use the four-step method and a tree diagram.

**STAFF NOTE:** Try to get a brief discussion going on the issue “How could knowing about the eight of hearts be relevant to the presence of the Ace of spades?”

**Solution.** Let \( C \) be the event that you pick the complete deck, and let \( H \) be the event that you select the eight of hearts. In these terms, our aim is to compute:
\[
\Pr[C \mid H] = \frac{\Pr[C \cap H]}{\Pr[H]}
\]

A tree diagram is worked out below:

Now we can compute the desired conditional probability as follows:

\[
\Pr[C \mid H] = \frac{\Pr[C \cap H]}{\Pr[H]} = \frac{\frac{1}{2} \cdot \frac{1}{52}}{\frac{1}{2} \cdot \frac{1}{52} + \frac{1}{2} \cdot \frac{1}{51}} = \frac{51}{103} = 0.495146 \ldots
\]

Thus, if you selected the eight of hearts, then the deck you picked is less likely to be the complete one. It’s worth stopping to think about how you might have arrived at this final conclusion without going through the detailed calculation—or better, how you might explain it to your 10-year-old niece.

The explanation is simple: drawing an eight of hearts from a small deck containing an eight of hearts is more likely than drawing one from a larger such deck. So if you see an eight of hearts, it’s more likely to have come from a smaller deck. The soundness of this intuitive explanation is proved in Problem ??.

**Supplemental problem**

**Problem 4.**

There is a subject—naturally not *Math for Computer Science*—in which 10% of the assigned problems contain errors. If you ask a Teaching Assistant (TA) whether a problem has an error, then they will answer correctly 80% of the time, regardless of whether or not a problem has an error. If you ask a lecturer, he will identify whether or not there is an error with only 75% accuracy.

We formulate this as an experiment of choosing one problem randomly and asking a particular TA and
Lecturer about it. Define the following events:

\[ E := \text{[the problem has an error]}, \]
\[ T := \text{[the TA says the problem has an error]}, \]
\[ L := \text{[the lecturer says the problem has an error]}. \]

(a) Translate the description above into a precise set of equations involving conditional probabilities among the events \( E, T, \) and \( L. \)

**Solution.** The assumptions above tell us:

\[
\Pr[E] = \frac{10}{100} = \frac{1}{10}, \quad \Pr[T \mid E] = \Pr[\overline{T} \mid E] = \frac{80}{100} = \frac{4}{5}, \quad \Pr[L \mid E] = \Pr[\overline{L} \mid E] = \frac{75}{100} = \frac{3}{4}.
\]

Also, \( T \) and \( L \) are independent given \( E, \) and given \( \overline{E}: \)

\[
\Pr[T \cap L \mid E] = \Pr[T \mid E] \Pr[L \mid E], \quad \Pr[T \cap L \mid \overline{E}] = \Pr[T \mid \overline{E}] \Pr[L \mid \overline{E}].
\]

Note that while we know that \( T \) and \( L \) are independent given \( E \) or given \( \overline{E}, \) they are not independent by themselves, see part (c).

(b) Suppose you have doubts about a problem and ask a TA about it, and they tell you that the problem is correct. To double-check, you ask a lecturer, who says that the problem has an error. Assuming that the correctness of the lecturers’ answer and the TA’s answer are independent of each other, regardless of whether there is an error, what is the probability that there is an error in the problem?

**Solution.** We want to calculate

\[ \Pr[E \mid \overline{T} \cap L]. \]

From the definition of conditional probability (this is known as Bayes’ rule):

\[
\Pr[E \mid \overline{T} \cap L] = \Pr[E] \cdot \frac{\Pr[\overline{T} \cap L \mid E]}{\Pr[\overline{T} \cap L]}.
\]

By the independence assumptions, we have:

\[
\Pr[\overline{T} \cap L \mid E] = \Pr[\overline{T} \mid E] \Pr[L \mid E] = \frac{13}{54} = \frac{3}{20}, \quad \Pr[T \cap L \mid \overline{E}] = \Pr[T \mid \overline{E}] \Pr[L \mid \overline{E}] = \frac{4}{5} \cdot \frac{1}{4} = \frac{1}{5},
\]

\[
\Pr[T \cap L] = \Pr[T \cap L \mid E] \Pr[E] + \Pr[T \cap L \mid \overline{E}] \Pr[\overline{E}] = \frac{3}{20} \cdot \frac{1}{10} + \frac{1}{5} \cdot \frac{9}{10} = \frac{39}{200}.
\]

Substituting these values in equation (1), we get

\[
\Pr[E \mid \overline{T} \cap L] = \frac{1}{10} \cdot \frac{3/20}{39/200} = \frac{1}{13} \approx 0.077.
\]
So this contradictory information has decreased the probability of an error from 10% to about 7.7%.

The calculations here support the common-sense rule that when two people make contradictory statements, you should be influenced more by the most "authoritative" person—the one who is right more often. But note that this does not mean that you should believe in what the most authoritative person says, since the probability of an error remains uncomfortably large.

(e) Is event $T$ independent of event $L$ (that is, $\Pr[T \mid L] = \Pr[T]$)?

**Solution.** The answer is no. Because the TA is usually right, when the TA says that the problem has an error, the likelihood that there really is an error is increased. But the lecturer is also usually right, so increasing the likelihood of there being an error also increases the likelihood that the lecturer will report an error.

We verify this informal argument by actually calculating the probability of each of these events and their conjunction, and observing that the probability that the two events occur is different from the product of the probabilities. Let events $E, T, L$ be as above.

\[
\Pr[T] = \Pr[T \cap E] + \Pr[T \cap \overline{E}] \\
= \Pr[T \mid E] \Pr[E] + \Pr[T \mid \overline{E}] \Pr[\overline{E}] \\
= \frac{4}{5} \cdot \frac{1}{10} + (1 - \frac{4}{5})(1 - \frac{1}{10}) = \frac{13}{50},
\]

\[
\Pr[L] = \Pr[L \cap E] + \Pr[L \cap \overline{E}] \\
= \frac{3}{4} \cdot \frac{1}{10} + (1 - \frac{3}{4})(1 - \frac{1}{10}) = \frac{3}{10},
\]

\[
\Pr[L \cap T] = \Pr[L \cap T \cap E] + \Pr[L \cap T \cap \overline{E}] \\
= \Pr[L \cap T \mid E] \Pr[E] + \Pr[L \cap T \mid \overline{E}] \Pr[\overline{E}] \\
= \Pr[L \mid E] \Pr[T \mid E] \Pr[E] + \Pr[L \mid \overline{E}] \Pr[T \mid \overline{E}] \Pr[\overline{E}] \\
= \frac{3}{4} \cdot \frac{4}{5} \cdot \frac{1}{10} + (1 - \frac{3}{4})(1 - \frac{4}{5}) \cdot (1 - \frac{1}{10}) = \frac{105}{1000} = 0.105,
\]

which is higher than

\[
\Pr[L] \Pr[T] = \frac{3}{10} \cdot \frac{13}{50} = 0.078.
\]