Uncountable Sets

Are all sets the same size? NO!

Cantor’s Theorem shows how to keep finding bigger infinities.

Countable Sets

A is countable iff can list it:

\[ a_0, a_1, a_2, \ldots \]

Example:

\( \{0, 1\}^\omega \) is \{finite bit strings\}

Claim: \( \{0, 1\}^\omega \) is uncountable.

Diagonal Arguments

Suppose \( s_0, s_1, s_2, \ldots \in \{0, 1\}^\omega \)

\[
\begin{array}{ccccccc}
0 & 1 & 2 & 3 & \cdots & n & n+1 & \cdots \\
S_0 & 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\
S_1 & 0 & 1 & 1 & 0 & \ldots & 0 & 1 \\
S_2 & 1 & 0 & 0 & 0 & \ldots & 1 & 0 \\
S_3 & 1 & 0 & 1 & 1 & \ldots & 1 & 1 \\
& & & & & \vdots & \vdots & \vdots \\
& & & & & 1 & 1 & 1 \\
& & & & & 0 & 0 & 0
\end{array}
\]
Suppose \( s_0, s_1, s_2, \ldots \in \{0,1\}^\omega \)

<table>
<thead>
<tr>
<th>s_0</th>
<th>0</th>
<th>1</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>s_1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>s_2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>s_3</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
</tbody>
</table>

So \( 1010001 \ldots \) differs from every row. That is,

\[
1010001 \ldots \not\in \{0,1\}^\omega
\]

contradicting the claim that every \( s \in \{0,1\}^\omega \) appears as a row.

\[\{0,1\}^\omega \text{ is uncountable}\]

...proving that \( \text{NOT} \left( \mathbb{N} \text{ surj} \{0,1\}^\omega \right) \)

also \( \{0,1\}^\omega \text{ surj} \mathbb{N} \) (easy)

making \( \mathbb{N} \) "strictly smaller" than \( \{0,1\}^\omega \)
Strictly Smaller

A strict B ::= NOT(A surj B)
A is “strictly smaller” than B

Cantor’s Theorem

A strict pow(A)
for every set, A
(finite or infinite)

A strict Pow(A)
Pf: say have fcn f:A→pow(A).
Define a subset of A that is not in
the range of f: namely
D ::= \{a ∈ A | a \not\in f(a)\}
Now D \not\in range(f) since it differs
from set f(a) at element a!

A strict Pow(A)
Pf: say have fcn f:A→pow(A).
a ∈ D \iff a \not\in f(a)
for all a ∈ A by def of D.
Suppose D ∈ range(f). That is
D = f(a_d) for some a_d ∈ A
Let $a$ be $a_d$.

A strict $\text{Pow}(A)$

Pf: say have fcn $f:A \rightarrow \text{pow}(A)$.

\[ a \in f(a_d) \iff a \notin f(a) \]
for all $a \in A$ by def of $D$, $a_d$.

Let $a$ be $a_d$.

\[ a_d \in f(a_d) \iff a_d \notin f(a_d) \]
for all $a \in A$ by def of $D$, $a_d$.

A strict $\text{Pow}(A)$

Pf: say have fcn $f:A \rightarrow \text{pow}(A)$.

Let $a$ be $a_d$.

A strict $\text{Pow}(A)$

So no $f$-arrow into $D$.

$f$ is not a surjection.

QED
\{0,1\}^\omega \text{ is uncountable by Cantor: }
\begin{align*}
N &\rightarrow \{0,1\}^\omega \\
&\rightarrow \text{pow}(N) \\
\end{align*}

\text{surj?}

\{0,1\}^\omega \text{ is uncountable by Cantor: }
\begin{align*}
N &\rightarrow \{0,1\}^\omega \\
&\rightarrow \text{pow}(N) \\
\end{align*}

\text{surj? bij (from before)}

\{0,1\}^\omega \text{ is uncountable by Cantor: }
\begin{align*}
N &\rightarrow \{0,1\}^\omega \\
&\rightarrow \text{pow}(N) \\
\end{align*}

\text{surj? bij}

\text{contradiction}