Set Theory

Self application

Self application is notoriously doubtful:

“This statement is false.”

is it true or false?

Self membership

The list

\[ L = (0 \ 1 \ 2) \]

\[
\begin{array}{ccc}
L & \rightarrow & 0 \\
& & 1 \\
& & 2 \\
\end{array}
\]
Self membership

(Lists are member of themselves:
\[ L = (0 \ (0 \ (0 \ldots 2) \ 2) \ 2) \]

compose procedures

\[
\begin{align*}
((\text{define}\ compose\ f\ g) \\
(\text{define}\ (h\ x)) \\
(f\ (g\ x)))

h)
\end{align*}
\]
compose procedures

$$((\text{compose square} \ \text{add1}) \ 3)$$
$$\Rightarrow \ 16 \ \ (= \ (3 + 1)^2)$$

$$((\text{compose square square}) \ 3)$$
$$\Rightarrow \ 81 \ \ (= \ (3^2)^2)$$

compose procedures

$$(\text{define comp2 f})$$
$$(\text{compose f f})$$

$$((\text{comp2 square}) \ 3)$$
$$\Rightarrow \ 81$$

apply procedure to itself:

$$(((\text{comp2 comp2}) \ \text{add1}) \ 3)$$
$$\Rightarrow \ 7$$

$$(((\text{comp2 comp2}) \ \text{square}) \ 3)$$
$$\Rightarrow \ 43046721 \ \ (= \ 3^{16})$$

Russell’s Paradox

Let $W := \{ s \in \text{Sets} \mid s \notin s \}$

so $[ s \in W \ \text{iff} \ s \notin s ]$

Now let $s$ be $W$, and reach a contradiction:

$[ W \in W \ \text{iff} \ W \notin W ]$
Disaster: Math is broken!

I am the Pope,
Pigs fly,
and verified programs crash...

...but paradox is buggy

Assumes that $W$ is a set!

$\left[ s \in W \iff s \notin s \right]$ for all sets $s$

...can only substitute $W$ for $s$ if $W$ is a set

...but paradox is buggy

Assumes that $W$ is a set!

We can avoid the paradox, if we deny that $W$ is a set!

...which raises the key question: just which well-defined collections are sets?

Zermelo-Frankel Set Theory

No simple answer, but the axioms of Zermelo-Frankel along with the Choice axiom (ZFC) do a pretty good job.
Some Axioms of Set Theory

Equality
\[ \forall x [x \in y \iff x \in z] \text{ IMPLIES } y = z \]

Power set
\[ \forall x \exists p \forall s. s \subseteq x \iff s \in p \]

Zermelo-Frankel Set Theory

According to ZF, the elements of a set have to be “simpler” than the set itself. In particular, no set is a member of itself, or a member of a member...

This implies that
(1) the collection of all sets is not a set, and
(2) \( W \) equals the collection of all sets ...which is why it’s not a set