Staff Solutions to Problem Set 4

Reading: Chapter 5.5.4. State Machines; Chapter 6. Recursive Data Types; Chapter 7. Infinite Sets

STAFF NOTE: Lectures covered: State Machines, Recursive Data, Infinite Sets

Problem 1.
A robot named Wall-E wanders around a two-dimensional grid. He starts out at (0, 0) and is allowed to take four different types of step:

1. (2, -1)
2. (1, -2)
3. (1, 1)
4. (-3, 0)

Thus, for example, Wall-E might walk as follows. The types of his steps are listed above the arrows.

\((0, 0) \rightarrow (2, -1) \rightarrow (3, 0) \rightarrow (4, -2) \rightarrow (1, -2) \rightarrow \ldots\)

Wall-E’s true love, the fashionable and high-powered robot, Eve, awaits at (0, 2).

(a) Describe a state machine model of this problem.

Solution. Let the set of states be \(\mathbb{Z} \times \mathbb{Z}\). The start state is (0, 0). The possible transitions are

\[(x, y) \rightarrow (x, y) + (u, v)\]  \hspace{1cm} (1)

for \((u, v) \in \{(+2, -1), (+1, -2), (-1, +1), (-3, 0)\}\).

(b) Will Wall-E ever find his true love? Either find a path from Wall-E to Eve or use the Invariant Principle to prove that no such path exists.

Solution. Let \(P(x, y)\) be the property that \(x + 2y\) is a multiple of 3. We claim \(P\) is a preserved invariant. To show this, we must show that if \(3 \mid x + 2y\), and Wall-E moves to \((x, y) + (u, v)\), then 3 divides

\[(x + u) + 2(y + v).\]  \hspace{1cm} (2)

But this value equals \(x + 2y + (u + 2v)\), \hspace{1cm} (3)

and since \(3 \mid u + 2v\)
for each of the four possible moves \((u, v)\) listed above (as is easily checked), we conclude that 3 divides both terms in the sum \((3)\) and therefore divides the whole sum. This proves implies that 3 divides \((2)\), completing the proof that \(P\) is preserved by transitions.

Now \(P\) holds in the start state, since \(3 \mid (0 + 2 \cdot 0)\). However, \(P\) does not hold for Eve’s position, \((0, 2)\), since \(0 + 2 \cdot 2 = 4\) is not a multiple of 3. Therefore, by the Invariant Principle, Eve’s position is not a reachable state.

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### Problem 2.

We’re going to characterize a large category of games as a recursive data type and then prove, by structural induction, a Fundamental Theorem about game strategies. The games we’ll consider are known as deterministic games of perfect information, because at each move, the complete game situation is known to the players, and this information completely determines how the rest of the game can be played. Games like chess, checkers, GO, and tic-tac-toe fit this description. In contrast, most card games do not fit, since card players usually do not know exactly what cards belong to the other players. Neither do games involving random features like dice rolls, since a player’s move does not uniquely determine what happens next.

Chess counts as a deterministic game of perfect information because at any point of play, both players know whose turn it is to move and the location of every chess piece on the board.\(^1\) At the start of the game, there are 20 possible first moves: the player with the White pieces can move one of his eight pawns forward 1 or 2 squares or one of his two knights forward and left or forward and right. For the second move, the Black player can make one of the 20 corresponding moves of his own pieces. The White player would then make the third move, but now the number of possible third moves depends on what the first two moves happened to be.

A nice way to think of these games is to regard each game situation as a game in its own right. For example, after five moves in a chess game, we think of the players as being at the start of a new “chess” game determined by the current board position and the fact that it is Black’s turn to make the next move.

At the end of a chess game, we might assign a score of 1 if the White player won, \(-1\) if White lost, and 0 if the game ended in a stalemate (a tie). Now we can say that White’s objective is to maximize the final score and Black’s objective is to minimize it. We might also choose to score the game in a more elaborate way, taking into account not only who won, but factors like how many moves the game took, or the final board configuration.

This leads to an elegant abstraction of this kind of game. We suppose there are two players, called the \textit{max-player} and the \textit{min-player}, whose aim is, respectively, to maximize and minimize the final score. A game will specify its set of possible first moves, each of which will simply be another game. A game with no possible moves is called an \textit{ended game}, and will just have a final score. Strategically, all that matters about an ended game is its score. If a game is not ended, it will have a label \(\max\) or \(\min\) indicating which player is supposed to move first.

This motivates the following formal definition:

\textbf{Definition.} Let \(V\) be a nonempty set of real numbers. The class \(\text{VG}\) of \(V\)-valued deterministic max-min games of perfect information is defined recursively as follows:

\textbf{Base case:} A value \(v \in V\) is a VG, and is called an \textit{ended game}.

\textbf{Constructor case:} If \(\{G_0, G_1, \ldots\}\) is a nonempty set of VG’s, and \(a\) is a label equal to \(\max\) or \(\min\), then

\[G := (a, \{G_0, G_1, \ldots\})\]

\(^1\)In order to prevent the possibility of an unending game, chess rules specify a limit on the number of moves, or a limit on the number of times a given board position may repeat. So the number of moves or the number of position repeats would count as part of the game situation known to both players.
is a VG. Each game $G_i$ is called a possible first move of $G$.

In all the games like this that we’re familiar with, there are only a finite number of possible first moves. It’s worth noting that the definition of VG does not require this. Since finiteness is not needed to prove any of the results below, it would arguably be misleading to assume it. Later, we’ll suggest how games with an infinite number of possible first moves might come up.

A play of a game is a sequence of legal moves that either goes on forever or finishes with an ended game. More formally:

**Definition.** A play of a game $G \in VG$ is defined recursively on the definition of VG:

**Base case:** ($G$ is an ended game.) Then the length one sequence $(G)$ is a play of $G$.

**Constructor case:** ($G$ is not an ended game.) Then a play of $G$ is a sequence that starts with a possible first move, $G_i$, of $G$ and continues with the elements of a play of $G_i$.

If a play does not go on forever, its payoff is defined to be the value it ends with.

Let’s first rule out the possibility of playing forever. Namely, every play will have a payoff.

**(a)** Prove that every play of a $G \in VG$ is a finite sequence that ends with a value in $V$. **Hint:** By structural induction on the definition of VG.

**Solution.** We prove by structural induction on $G \in VG$ that every play of $G$ is finite and ends with a value.

**Base case:** [$G$ is the ended game $v \in V$.] Then there is only one play of $G$, namely the length one play $(v)$, which certainly ends with a value.

**Constructor case:** [$G = (a, \{G_0, G_1, \ldots\})$.] A play of $G$ by definition consists of a sequence that starts with some first move $G_j$ and continues with a play of $G_j$. By structural induction, we know that this play of $G_j$ is a sequence of some finite length $n$ that ends with a value, so this play of $G$ is a length $n + 1$ sequence that ends with the same value. ■

A strategy for a game is a rule that tells a player which move to make when it’s his turn. Formally:

**Definition.** If $a$ is one of the labels $\text{max}$ or $\text{min}$, then an $a$-strategy is a function $s : VG \rightarrow VG$ such that

$s(G)$ is \[
\begin{cases} 
\text{a first move of } G & \text{if } G \text{ has label } a, \\
\text{undefined} & \text{otherwise.}
\end{cases}
\]

Any pair of strategies for the two players determines a unique play of a game, and hence a unique payoff, in an obvious way. Namely, when it is a player’s turn to move in a game $G$, he chooses the move specified by his strategy. A strategy for the max-player is said to ensure payoff $v$ when, paired with any strategy for the min-player, the resulting payoff is at least $v$. Dually, a strategy for the min-player caps payoff at $v$ when, paired with any strategy for the max-player, the resulting payoff is at most $v$.

Assuming for simplicity that the set $V$ of possible values of a game is finite, the WOP (Section 2.4) implies there will be a strategy for the max-player that ensures the largest possible payoff; this is called the max-ensured-value of the game. Dually, there will also be a strategy for the min-player that caps the payoff at the smallest possible value, which is called the min-capped-value of the game.

The max-ensured-value of course cannot be larger than the min-capped-value. A unique value can be assigned to a game when these two values agree:

**Definition.** If the max-ensured-value and min-capped-value of a game are equal, their common value is called the value of the game.
So if both players play optimally in a game with that has a value, $v$, then there is actually no point in playing. Since the payoff is ensured to be at least $v$ and is also capped to be at most $v$, it must be exactly $v$. So the min-player may as well skip playing and simply pay $v$ to the max-player (a negative payment means the max-player is paying the min-player).

The punch line of our story is that the max-ensured-value and the min-capped-value are always equal.

Theorem (Fundamental Theorem for Deterministic Min-Max Games of Perfect Information). Let $V$ be a finite set of real numbers. Every $V$-valued deterministic max-min game of perfect information has a value.

(b) Prove this Fundamental Theorem for VG’s by structural induction.

Solution. The proof is by structural induction on the definition of a $G \in \text{VG}$. The induction hypothesis is that there is that

$$G$$’s max-ensured-value equals $G$’s min-capped-value.

Base case: [$G$ is the ended game $v \in V$.] The only possible play is $(v)$. So the max-ensured-value and the min-capped-value are both $v$.

Constructor case: [$G = (a, \{G_0, G_1, \ldots\})$]. By structural induction we may assume that the max-ensured-value and min-capped-value are equal for each of the possible first moves $G_i$.

Case ($a = \text{max}$.) Let $v$ be the maximum of the values of the first moves of $G$. This max will exist because $V$ is assumed to be finite.

Now a strategy for the max-player that ensures $v$ is:

Choose a first move, $G_i$, that has the maximum value $v$, and then follow the strategy that ensures $v$ for $G_i$.

Dually, a strategy whereby the min-player can cap payoff at $v$ is:

Let $G_j$ be whatever first move is chosen by the max-player, and let $w$ be the value of $G_j$. Now follow the strategy in $G_j$ that caps the value at $w$. But $w \leq v$ since $v$ is the maximum value among the first moves, so this strategy will also cap the value at $v$.

So $v$ can be both ensured and capped at, and hence it is the value of $G$.

Case ($a = \text{min}$): Let $v$ be the minimum of the values of the next moves of $G$. This minimum exists for the same reasons as in the previous case.

Now a strategy for the min-player to cap payoff to $v$ is to choose a first move, $G_i$, with value $v$, and then to follow the strategy in $G_i$ that caps the payoff at $v$.

Dually, a strategy whereby the max-player can ensure payoff $v$ is to let $G_j$ be whatever first move is chosen by the min player and then to follow the strategy in $G_j$ that ensures the value of $G_j$, which is at least $v$.

So $v$ can be both capped at and ensured, and hence is the value of $G$.

So in any case, the game $G$ has a value, which completes the constructor case of the structural induction.

(c) Conclude immediately that in chess, there is a winning strategy for White, or a winning strategy for Black, or both players have strategies that guarantee at least a stalemate. (The only difficulty is that no one knows which case holds.)

Solution. By the fundamental theorem, the value of chess must be 1, $-1$, or 0.
So where do we come upon games with an infinite number of first moves? Well, suppose we play a tournament of \( n \) chess games for some positive integer \( n \). This tournament will be a VG if we agree on a rule for combining the payoffs of the \( n \) individual chess games into a final payoff for the whole tournament.

There still are only a finite number of possible moves at any stage of the \( n \)-game chess tournament, but we can define a meta-chess-tournament, whose first move is a choice of any positive integer \( n \), after which we play an \( n \)-game tournament. Now the meta-chess-tournament has an infinite number of first moves.

Of course only the first move in the meta-chess-tournament is infinite, but then we could set up a tournament consisting of \( n \) meta-chess-tournaments. This would be a game with \( n \) possible infinite moves. And then we could have a meta-meta-chess-tournament whose first move was to choose how many meta-chess-tournaments to play. This meta-meta-chess-tournament will have an infinite number of infinite moves. Then we could move on to meta-meta-meta-chess-tournaments . . . .

As silly or weird as these meta games may seem, their weirdness doesn’t disqualify the Fundamental Theorem: each of these games will still have a value.

(d) State some reasonable generalization of the Fundamental Theorem to games with an infinite set \( V \) of possible payoffs. Optional: Prove your generalization.

Solution. The obvious generalization would redefine the max-value as the lub of the ensured values, and the min-value as the glb of the limits to payoffs. The result is that some games may now have a value \( v \) that is positive or negative infinity, and that \( v \) can’t exactly be ensured or limted to, but rather that for any \( \epsilon > 0 \) there will be a strategy that ensures a value of at least \( v - \epsilon \) and a strategy that limits payoff to at most \( v + \epsilon \).

Problem 3.
In this problem you will prove a fact that may surprise you—or make you even more convinced that set theory is nonsense: the half-open unit interval is actually the same size as the nonnegative quadrant of the real plane!\(^2\) Namely, there is a bijection from \((0, 1]\) to \([0, \infty) \times [0, \infty)\).

(a) Describe a bijection from \((0, 1]\) to \([0, \infty)\).

Hint: \(1/x\) almost works.

Solution. \(f(x) := 1/x\) defines a bijection from \((0, 1]\) to \([1, \infty)\), so \(g(x) := f(x) - 1\) does the job.

(b) An infinite sequence of the decimal digits \(\{0, 1, \ldots, 9\}\) will be called long if it has infinitely many occurrences of some digit other than 0. Let \(L\) be the set of all such long sequences. Describe a bijection from \(L\) to the half-open real interval \((0, 1]\).

Hint: Put a decimal point at the beginning of the sequence.

Solution. Putting a decimal point in front of a long sequence defines a bijection from \(L\) to \((0, 1]\). This follows because every real number in \((0, 1]\) has a unique long decimal expansion. Note that if we didn’t exclude the non-long sequences, namely, those sequences ending with all zeroes, this wouldn’t be a bijection. For example, the sequences \(1000\ldots\) and \(099999\ldots\) would both map to the same real number, namely, \(1/10\).

(c) Describe a surjective function from \(L\) to \(L^2\) that involves alternating digits from two long sequences.

Hint: The surjection need not be total.

\(^2\)The half open unit interval, \((0, 1]\), is \(\{r \in \mathbb{R} \mid 0 < r \leq 1\}\). Similarly, \([0, \infty) := \{r \in \mathbb{R} \mid r \geq 0\}\).
Solution. Given any long sequence \( s = x_0, x_1, x_2, \ldots \), let
\[
h_0(s) := x_0, x_2, x_4, \ldots
\]
be the sequence of digits in even positions. Similarly, let
\[
h_1(s) := x_1, x_3, x_5, \ldots
\]
be the sequence of digits in odd positions. Then \( h \) is a surjective function from \( L \) to \( L^2 \), where
\[
h(s) := \begin{cases} 
(h_1(s), h_2(s)), & \text{if } h_1(s) \in L \text{ and } h_2(s) \in L, \\
\text{undefined}, & \text{otherwise}.
\end{cases}
\] (4)

(d) Prove the following lemma and use it to conclude that there is a bijection from \( L^2 \) to \( (0, 1]^2 \).

Lemma 3.1. Let \( A \) and \( B \) be nonempty sets. If there is a bijection from \( A \) to \( B \), then there is also a bijection from \( A \times A \) to \( B \times B \).

Solution. Proof. Suppose \( f : A \to B \) is a bijection. Let \( g : A^2 \to B^2 \) be the function defined by the rule \( g(x, y) = (f(x), f(y)) \). It is easy to show that \( g \) is a bijection:

- \( g \) is total: Since \( f \) is total, \( f(a_1) \) and \( f(a_2) \) exist \( \forall a_1, a_2 \in A \) and so \( g(a_1, a_2) = (f(a_1), f(a_2)) \) also exists.

- \( g \) is surjective: Since \( f \) is surjective, for any \( b_i \in B \) there exists \( a_i \in A \) such that \( b_i = f(a_i) \). So for any \((b_1, b_2)\) in \( B^2 \), there is a pair \((a_1, a_2)\) in \( A^2 \) such that \( g(a_1, a_2) := (f(a_1), f(a_2)) = (b_1, b_2) \). This shows that \( g \) is a surjection.

- \( g \) is injective:

\[
g(a_1, a_2) = g(a_3, a_4) \iff (f(a_1), f(a_2)) = (f(a_3), f(a_4)) \tag{by def of g}
\]
\[
\quad \iff f(a_1) = f(a_3) \text{ AND } f(a_2) = f(a_4)
\]
\[
\quad \iff a_1 = a_3 \text{ AND } a_2 = a_4(\text{since } f \text{ is injective})
\]
\[
(a_1, a_2) = (a_3, a_4).
\]

which confirms that \( g \) is injective.

Since it was shown in part (b) that there is a bijection from \( L \) to \( (0, 1] \), an immediate corollary of the Lemma is that there is a bijection from \( L^2 \) to \( (0, 1]^2 \).

(e) Conclude from the previous parts that there is a surjection from \( (0, 1] \) and \( (0, 1]^2 \). Then appeal to the Schröder-Bernstein Theorem to show that there is actually a bijection from \( (0, 1] \) and \( (0, 1]^2 \).

Solution. There is a bijection between \( (0, 1] \) and \( L \) by part (b), a surjective function from \( L \) to \( L^2 \) by part (c), and a bijection from \( L^2 \) to \( (0, 1]^2 \) by part (d). These surjections compose to yield a surjection from \( (0, 1] \) to \( (0, 1]^2 \).

Conversely, there is obviously a surjective function \( f : (0, 1]^2 \to (0, 1] \), namely
\[
f((x, y)) := x.
\]

The Schröder-Bernstein Theorem now implies that there is a bijection from \( (0, 1] \) to \( (0, 1]^2 \).
(f) Complete the proof that there is a bijection from \((0, 1]\) to \([0, \infty)^2\).

**Solution.** There is a bijection from \((0, 1]\) to \((0, 1]^2\) by part (e), and there is a bijection from \((0, 1]^2\) to \([0, \infty)^2\) by part (a) and the Lemma. These bijections compose to yield a bijection from \((0, 1]\) to \([0, \infty)^2\). ■