Staff Solutions to Problem Set 2

Reading: Chapter 3. Logical Formulas; Chapter 4. Mathematical Data Types through 4.4. Binary Relations.

STAFF NOTE: Lectures covered: Propositional Logic, Predicate Formulas, Sets & Relations

Problems should be submitted separately following the pset submission instructions, and each problem should have an attached collaboration statement.

Problem 1.
Translate the following sentence into a predicate formula:

There is a student who has emailed exactly two other people in the class, besides possibly herself.

The domain of discourse should be the set of students in the class; in addition, the only predicates that you may use are

- equality, and
- \( E(x, y) \), meaning that “\( x \) has sent e-mail to \( y \).”

Solution. A good way to begin tackling this problem is by working “top-down” to translate the successive parts of the sentence. First of all, our formula must be of the form

\[
\exists x. P(x)
\]

where \( P(x) \) should be a formula that says that “\( x \) has e-mailed exactly two other people in the class, besides possibly herself”.

One way to write \( P(x) \) is to give names, say \( y \) and \( z \), to the two students whom \( x \) has emailed. So we translate \( P(x) \) as “besides \( x \), there are two students, \( y \) and \( z \), and . . . ”:

\[
\exists y, z. x \neq y \land x \neq z \land y \neq z \land \ldots
\]

“\( x \) has emailed both \( y \) and \( z \), and . . . ”:

\[
E(x, y) \land E(x, z) \land \ldots
\]

“if \( x \) has emailed somebody, it’s either \( x \), \( y \), or \( z \).”:

\[
\forall s. E(x, s) \rightarrow (s = x \lor s = y \lor s = z).
\]

Putting these together, we get:

\[
P(x) := \exists y, z. \ x \neq y \land x \neq z \land y \neq z \land
E(x, y) \land E(x, z) \land
[\forall s. E(x, s) \rightarrow (s = x \lor s = y \lor s = z)]
\]
Problem 2.
A certain cabal within the Math for Computer Science course staff is plotting to make the final exam *ridiculously hard.* ("Problem 1. Prove the Poincare Conjecture starting from the axioms of ZFC. Express your answer in khipu—the knot language of the Incas.") The only way to stop their evil plan is to determine exactly who is in the cabal. The course staff consists of seven people:

\{Giuliano, Uddhav, Albert, Sharon, Sebastian, Nirvan, Eka\}

The cabal is a subset of these seven. A membership roster has been found and appears below, but it is deviously encrypted in logic notation. The predicate cabal indicates who is in the cabal; that is, cabal(x) is true if and only if x is a member. Translate each statement below into English and deduce who is in the cabal.

(a) \( \exists x, y, z. (x \neq y \text{ AND } x \neq z \text{ AND } y \neq z \text{ AND } \text{cabal}(x) \text{ AND } \text{cabal}(y) \text{ AND } \text{cabal}(z)) \)

**Solution.** A direct English paraphrase would be “There exist people we’ll call x, y, and z, who are all different, such that x, y and z are each in the cabal.” A better version would use the fact that there’s no need in this case to give names to the people. Namely, a better paraphrase is “There are at least 3 different people in the cabal.” Perhaps a simpler way to say this is: “The cabal is of size at least 3.”

(b) \( \neg(\text{cabal}(Eka) \text{ AND } \text{cabal}(Sharon)) \)

**Solution.** Eka and Sharon are not both in the cabal. Equivalently: at least one of Eka and Sharon is not in the cabal.

(c) \( \text{cabal}(Nirvan) \text{ IMPLIES } \forall x. \text{cabal}(x) \)

**Solution.** If Nirvan is in the cabal, then everyone is.

(d) \( \text{cabal}(Sharon) \text{ IMPLIES } \text{cabal}(Eka) \)

**Solution.** If Sharon is in the cabal, then Eka is also.

(e) \( (\text{cabal}(Sebastian) \text{ OR } \text{cabal}(Albert)) \text{ IMPLIES } \neg(\text{cabal}(Uddhav)) \)

**Solution.** If either of Sebastian or Albert is in the cabal, then Uddhav is not. Equivalently, if Uddhav is in the cabal, the neither Albert nor Sebastian is.

(f) \( (\text{cabal}(Sebastian) \text{ OR } \text{cabal}(Eka)) \text{ IMPLIES } \neg(\text{cabal}(Giuliano)) \)

**Solution.** If either of Sebastian or Eka is in the cabal, then Giuliano is not. Equivalently, if Giuliano is in the cabal, the neither Sebastian nor Eka is.

(g) Now use these facts to figure out exactly who is on the cabal.

**STAFF NOTE:** If a team is stuck, tell them that the cabal consists of exactly Eka, Sebastian, and Albert and have them check that this set satisfies all the conditions. (See the end of the solution.) Then start them back on proving that this is the unique set that works.
**Solution.** So much for the translations. We now argue that the only cabal satisfying all six propositions above is one whose members are exactly Eka, Sebastian, and Albert.

We first observe that by (b), there must be someone—either Eka or Sharon—who is not in the cabal. But if Nirvan were in the cabal, then by (c), everyone would be. So we conclude by contradiction that:

\[
\text{Nirvan is not in the cabal.}
\] (1)

Next observe that if Sharon was in the cabal, then by (d), Eka would be too, contradicting (b). So by again contradiction, we conclude:

\[
\text{Sharon is not in the cabal.}
\] (2)

Now suppose Uddhav is in the cabal. Then by (e), Sebastian and Albert are not, and we already know Nirvan and Sharon are not, so only three remain who could be in the cabal, namely, Uddhav, Eka, and Giuliano. But by (a) the cabal must have at least three members, so it follows that the cabal must consist of exactly these three. This proves:

**Lemma 2.1 (TNE).** If Uddhav is in the cabal, then Eka and Giuliano are in the cabal.

But by (f), if Eka is in the cabal, then Giuliano is not. That is,

**Lemma 2.2 (NnE).** Eka and Giuliano cannot both be in the cabal.

Now from Lemma (NnE) we conclude that the conclusion of Lemma (TNE) is false. So by contrapositive, the hypothesis of Lemma (TNE) must also be false, namely,

\[
\text{Uddhav is not in the cabal.}
\] (3)

Finally, suppose Giuliano is in the cabal. Then by (f), Sebastian and Eka are not, and we already know Nirvan, Sharon and Uddhav are not. So the cabal must consist of at most two people (Albert and Giuliano). This contradicts (a), and we conclude by contradiction that

\[
\text{Giuliano is not in the cabal.}
\] (4)

So the only remaining people who could be in the cabal are Albert, Sebastian, and Eka. Since the cabal must have at least three members, we conclude that

**Lemma.** The only possible cabal consists of Albert, Sebastian, and Eka.

But we’re not done yet: we haven’t shown that a cabal consisting of Albert, Sebastian, and Eka actually does satisfy all six conditions. So let \( \mathcal{A} = \{\text{Albert, Sebastian, Eka}\} \), and let’s quickly check that \( \mathcal{A} \) satisfies (a)–(f):

- \( |\mathcal{A}| = 3 \), so \( \mathcal{A} \) satisfies (a).
- Sharon is not in \( \mathcal{A} \), so \( \mathcal{A} \) satisfies (b) and (d).
- Nirvan is not in \( \mathcal{A} \), so the hypothesis of (c) is false, which means that \( \mathcal{A} \) satisfies (c).
- Finally, Uddhav and Giuliano are not in \( \mathcal{A} \), so the conclusions of both (e) and (f) are true, and so \( \mathcal{A} \) satisfies (e) and (f).

So now we have proved

**Proposition.** \( \{\text{Albert, Sebastian, Eka}\} \) is the unique cabal satisfying conditions (a)–(f).
Problem 3. (a) Give an example where the following result fails:

**False Theorem.** For sets $A$, $B$, $C$, and $D$, let

$$L := (A \cup B) \times (C \cup D),$$

$$R := (A \times C) \cup (B \times D).$$

Then $L = R$.

**Solution.** If $A = D = \emptyset$ and $B$ and $C$ are both nonempty, then $L = B \times C \neq \emptyset$, but $R = \emptyset$.

(b) Identify the mistake in the following proof of the False Theorem.

*Bogus proof.* Since $L$ and $R$ are both sets of pairs, it’s sufficient to prove that $(x, y) \in L \iff (x, y) \in R$ for all $x, y$.

The proof will be a chain of iff implications:

- $(x, y) \in R$
- $(x, y) \in (A \times C) \cup (B \times D)$
- $(x, y) \in A \times C$, or $(x, y) \in B \times D$
- $(x \in A$ and $y \in C$) or else $(x \in B$ and $y \in D)$
- either $x \in A$ or $x \in B$, and either $y \in C$ or $y \in D$
- $x \in A \cup B$ and $y \in C \cup D$
- $(x, y) \in L$.

**Solution.** The mistake is in the fourth “iff.” If $[x \in A$ or $x \in B$, and either $y \in C$ or $y \in D]$, it does not necessarily follow that $[(x \in A$ and $y \in C$) or else $(x \in B$ and $y \in D)]$. It might be that $(x, y)$ is in $A \times D$ instead. This happens, for example, if $A = \{1\}$, $B = \{2\}$, $C = \{3\}$, $D = \{4\}$, and $(x, y) = (1, 4)$.

(c) Fix the proof to show that $R \subseteq L$.

**Solution.** Replacing the fourth “iff” with “implies that” yields a correct proof that $(x, y) \in R$ leads to $(x, y) \in L$, which implies that $R \subseteq L$.

Problem 4.

Let $A$, $B$, and $C$ be sets, and let $f : B \to C$ and $g : A \to B$ be functions. Let $h : A \to C$ be the composition, $f \circ g$, that is, $h(x) := f(g(x))$ for $x \in A$. Prove or disprove the following claims:

*Hint:* Arguments based on “arrows” using Definition 4.4.2 are fine.

(a) If $h$ is surjective, then $f$ must be surjective.

**Solution.** *True.*

For all $x$ in $C$: Since $h$ is surjective, there exists $y$ in $A$ such that $h(y) = x$. Therefore, by definition of $h$, $f(g(y)) = x$, so $x$ is in the range of $f$.

Therefore, all of $C$ is in the image of $f(C)$, so $f$ is surjective.

(b) If $h$ is surjective, then $g$ must be surjective.
Solution. False.
Suppose $A = C = \{1\}$ and $B = \{1, 2\}$. Let $f$ be such that $f(1) = f(2) = 1$, and $g$ such that $g(1) = 1$. In this case $h$ is indeed surjective, as $h(1) = 1$, but $g$ is not surjective as it doesn’t map anything to 2.

(c) If $h$ is injective, then $f$ must be injective.

Solution. False.
Taking the same example as in the previous case. $h$ is injective, because only 1 maps to 1. However, $f$ is not injective as $f(1) = f(2)$.

(d) If $h$ is injective and $f$ is total, then $g$ must be injective.

Solution. True.
For all $x$ and $y$: If $g(x) = g(y)$ then since $f$ is total, $f$ is defined on $g(x)$ and

$$h(x) = f(g(x)) = f(g(y)) = h(y),$$

so $x = y$ because $h$ is injective. This means, $g$ is injective.

Note that $g$ need not be injective when $f$ is not total (see Problem 4.16).