Staff Solutions to Problem Set 11


Problem 1. (a) Color each point in the plane with integer coordinates either red, white or blue. Let \( R \) be a 4 \( \times \) 82 rectangular grid of these points. Explain why at least two of the 82 rows in \( R \) must be colored the same.

Solution. There are only \( 3^4 = 81 \) ways to color the points in a row of length 4, and there are 82 rows, so by the Pigeonhole Principle, two rows must be colored the same.

(b) Conclude that \( R \) contains four points with the same color that form the corners of a rectangle.

Solution. Since each length 4 row of \( R \) is colored with only 3 colors, some color, say red, must occur more than once in each of the two rows with the same coloring. The first two points colored red in each of these two rows form a rectangle.

(c) Generalize the above argument to a coloring using the rainbow colors Red, Orange, Yellow, Green, Blue, Indigo, Violet along with White and Black.

Solution. There are now 9 colors, so a row of 10 points will have a color repeated, say red, and a rectangle with \( 9^{10} + 1 \) rows will have two rows with matching colors, so again, the first two red points in each of the matched rows will form a rectangle.

Problem 2.
How many paths are there from point (0, 0) to (50, 50) if each step along a path increments one coordinate and leaves the other unchanged? How many are there when there are impassable boulders sitting at points (10, 11) and (21, 20)? (You do not have to calculate the number explicitly; your answer may be an expression involving binomial coefficients.)

Hint: Inclusion-Exclusion.

STAFF NOTE: Hint: Suggest counting the number of paths going through (10, 11) and through both (10, 11) and (21, 20).

Solution. We use Inclusion-Exclusion. The total number of paths is \( \binom{100}{50} \), but we must subtract off the obstructed paths. There are \( \binom{21}{10} \cdot \binom{79}{40} \) paths through the first boulder, since there are \( \binom{21}{10} \) paths from the start to the first boulder and \( \binom{79}{40} \) paths from the boulder to the finish. Similarly, there are \( \binom{41}{20} \cdot \binom{59}{30} \) paths through the second boulder. However, we must subtract off paths going through both boulders. The number of these
is the number of path from origin to the first boulder times the number of paths from the first boulder to the second boulder times the number of paths from the second boulder to the end, namely
\[
\binom{21}{10} \cdot \binom{20}{9} \cdot \binom{59}{30}.
\]

Therefore, the total number of paths is:
\[
\binom{100}{50} - \binom{21}{10} \cdot \binom{79}{40} - \binom{41}{20} \cdot \binom{59}{30} + \binom{20}{10} \cdot \binom{20}{9} \cdot \binom{59}{30}
\]

Problem 3.

(a) Give a combinatorial proof of the following identity by letting $S$ be the set of all length-$n$ sequences of letters $a$, $b$ and a single $c$ and counting $|S|$ is two different ways.

\[
n2^{n-1} = \sum_{k=1}^{n} k \binom{n}{k} \tag{1}
\]

**Solution.** Let $P := [0, n] \times \{a, b\}^{n-1}$. On the one hand, there is a bijection from $P$ to $S$ by mapping $(k, x)$ to the word obtained by inserting a $c$ just after the $k$th letter in the length-$n-1$ word, $x$, of a’s and b’s. So
\[
|S| = |P| = n2^{n-1} \tag{2}
\]

by the Product Rule.

On the other hand, every sequence in $S$ contains between 1 and $n$ entries not equal to $a$ since the $c$, at least, is not $a$. The mapping from a sequence in $S$ with exactly $k$ non-$a$ entries to a pair consisting of the set of positions of the non-$a$ entries and the position of the $c$ among these entries is a bijection, and the number of such pairs is $\binom{n}{k}k$ by the Generalized Product Rule. Thus, by the Sum Rule:
\[
|S| = \sum_{k=1}^{n} k \binom{n}{k}
\]

Equating this expression and the expression (2) for $|S|$ proves the theorem. \[\blacksquare\]

(b) Now prove (1) algebraically by applying the Binomial Theorem to $(1 + x)^n$ and taking derivatives.

**Solution.** By the Binomial Theorem
\[
(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.
\]

Taking derivatives, we get
\[
n(1 + x)^{n-1} = \sum_{k=0}^{n} k \binom{n}{k} x^{k-1} = \frac{1}{x} \sum_{k=0}^{n} \binom{n}{k} x^k. \tag{3}
\]
Letting $x = 1$ in (3) yields (1).

Problem 4.
Miss McGillicuddy never goes outside without a collection of pets. In particular:

- She brings a positive number of songbirds, which always come in pairs.
- She may or may not bring her alligator, Freddy.
- She brings at least 2 cats.
- She brings two or more chihuahuas and labradors leashed together in a line.

Let $P_n$ denote the number of different collections of $n$ pets that can accompany her, where we regard chihuahuas and labradors leashed up in different orders as different collections, even if there are the same number chihuahuas and labradors leashed in the line.

For example, $P_6 = 4$ since there are 4 possible collections of 6 pets:

- 2 songbirds, 2 cats, 2 chihuahuas leashed in line
- 2 songbirds, 2 cats, 2 labradors leashed in line
- 2 songbirds, 2 cats, a labrador leashed behind a chihuahua
- 2 songbirds, 2 cats, a chihuahua leashed behind a labrador

And $P_7 = 16$ since there are 16 possible collections of 7 pets:

- 2 songbirds, 3 cats, 2 chihuahuas leashed in line
- 2 songbirds, 3 cats, 2 labradors leashed in line
- 2 songbirds, 3 cats, a labrador leashed behind a chihuahua
- 2 songbirds, 3 cats, a chihuahua leashed behind a labrador
- 4 collections consisting of 2 songbirds, 2 cats, 1 alligator, and a line of 2 dogs
- 8 collections consisting of 2 songbirds, 2 cats, and a line of 3 dogs.

(a) Let

$$P(x) := P_0 + P_1x + P_2x^2 + P_3x^3 + \cdots$$

be the generating function for the number of Miss McGillicuddy’s pet collections. Verify that

$$P(x) = \frac{4x^6}{(1 - x)^2(1 - 2x)}.$$
Solution.

\[ P(x) = \left( \frac{x^2 + x^4 + x^6 + x^8 + \cdots}{1 + x} \right) \]

\[ \cdot \left( \frac{x^2 + x^3 + x^4 + \cdots}{1 + x} \right) \]

\[ \cdot \left( \frac{2x^2 + 2^3x^3 + 2^4x^4 + \cdots}{1 + x} \right) \]

\[ \frac{\text{collections of songbirds}}{\text{collections of gators}} \cdot \frac{\text{collections of cats}}{\text{lines of dogs}} \]

= \frac{x^2}{1-x^2} \cdot \frac{x^2}{1-x} \cdot \frac{4x^2}{1-2x} \cdot \frac{x^2}{(1-x)(1+x)} \cdot \frac{x^2}{1-x} \cdot \frac{4x^2}{1-2x} \cdot \frac{4x^6}{1-2x^2} \cdot \frac{1}{(1-x)^2(1-2x)}. \]

(b) Find a simple formula for \( P_n \).

**Solution.** \( P_n \) is the coefficient of \( x^n \) in the power series for \( 4x^6/(1-x)^2(1-2x) \), which means it is 4 times the coefficient of \( x^{n-6} \) in the series for \( 1/(1-x)^2(1-2x) \) when \( n \geq 6 \), and \( P_n = 0 \) for \( n < 6 \).

But we can express \( 1/(1-x)^2(1-2x) \) using partial fractions as

\[ \frac{1}{(1-x)^2(1-2x)} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1-2x} \]  \hspace{1cm} (4)

for some constants \( A, B, C \), so \( P_n \) will be 4 times the sum of the coefficients of \( x^{n-6} \) in each of \( A/(1-x) \), \( B/(1-x)^2 \), and \( C/(1-2x) \), namely

\[ P_n = 4A + 4B(n - 5) + C2^{n-6}. \hspace{1cm} (5) \]

So we need only find the values of \( A, B, C \). But multiplying both sides of (4) by the lefthand denominator \((1-x)^2(1-2x)\) yields

\[ 1 = A(1-x)(1-2x) + B(1-2x) + C(1-x)^2. \hspace{1cm} (6) \]

Now letting \( x = 1 \) in (6) gives \( B = -1 \). Similarly, letting \( x = 1/2 \) gives \( C = 4 \). Finally, letting \( x = 0 \) gives \( A + B + C = 1 \) and so \( A = -2 \). Substituting these values into (5) finally gives

\[ P_n = 4(-2) - 4(n - 5) + 4(2^{n-4}) = 2^{n-2} - 4n + 12. \hspace{1cm} \]

Problem 5.

Let \( x_0 \equiv 0, x_1 \equiv 1 \) and for \( n \geq 2 \), let \( x_n \) be defined by the linear recurrence:

\[ x_n = 3x_{n-1} - 2x_{n-2} + n. \]

Find a closed form expression for \( x_n \).
Solution. We begin by finding the generating function for the sequence as defined by the recurrence:

\[ X(z) := x_0 + x_1 z + x_2 z^2 + \cdots + x_n z^n + \cdots. \]

We can do this by breaking the sequence into a sum of three sequences:

\[ X(z) = 3z X(z) - 2z^2 X(z) + \frac{z}{(1-z)^2} \]

So

\[ X(z) \cdot (1 - 3z + 2z^2) = \frac{z}{(1-z)^2} \]

Since \((1 - 3z + 2z^2) = (1 - 2z)(1 - z)\), we get:

\[ X(z) = \frac{z}{(1-z)^3(1-2z)} \]

Now, to find the closed form for the \(n\)th coefficient of this generating function, let’s expand \(X(z)\) into partial fractions:

\[ \frac{z}{(1-z)^3(1-2z)} = \frac{A}{(1-z)^3} + \frac{B}{(1-z)^2} + \frac{C}{1-z} + \frac{D}{1-2z} \]

To find the constants \(A, B, C, D\), we can multiply both sides by the denominator \((1-z)^3(1-2z)\), so

\[ z = A(1 - 2z) + B(1 - z)(1 - 2z) + C(1 - z)^2(1 - 2z) + D(1 - z)^3 \]

Now letting \(z = 1\), yields \(A = -1\), and letting \(z = 1/2\) yields \(1/2 = D/2^3\), that is, \(D = 4\). Then letting \(z = 0\) then yields \(B + C = -3\), and letting \(z = 2\) yields \(B - C = 1\), so \(B = -1\) and \(C = -2\).

So

\[ X(z) = \frac{z}{(1-z)^3(1-2z)} = -\frac{1}{(1-z)^3} - \frac{1}{(1-z)^2} - \frac{2}{1-z} + \frac{4}{1-2z}. \]

We know the coefficient of \(z^n\) in the power series for each term in this partial fraction expansion:

\[ -\frac{1}{(1-z)^3} = -(1 + z + \cdots + \left(\frac{n+2}{2}\right)z^n + \cdots) \]

\[ -\frac{1}{(1-z)^2} = -(1 + z + \cdots + \left(\frac{n+1}{1}\right)z^n + \cdots) \]

\[ -\frac{2}{1-z} = -2(1 + z + \cdots + z^n + \cdots) \]

\[ \frac{4}{1-2z} = 4(1 + z + \cdots + 2^n z^n + \cdots) \]

Summing up the coefficients of \(z^n\) in each of these power series gives:

\[ x_n = -\frac{(n+1)(n+2)}{2} - (n+1) - 2 + 4 \cdot 2^n \]

\[ = -\left(\frac{n^2}{2} + \frac{3n}{2} + 1\right) - n - 1 + 2 + 4 \cdot 2^n \]

\[ = 2^{n+2} - \frac{n^2}{2} - \frac{5n}{2} - 4. \]