Staff Solutions to Problem Set 1

Reading: Part I. Proofs: Introduction; Chapter 1. What is a Proof?; Chapter 2. The Well Ordering Principle. These assigned readings do not include the Problem sections. (Many of the problems in the text will appear as class or homework problems.)

Reminders:

- You are required to post some comments on the assigned reading using the class NB annotation system by the end of the second week of class (Feb. 15). After that commenting is encouraged but optional.

- The class also has a Piazza forum structured to get you help fast and efficiently from classmates and staff. With Piazza you may post questions—both administrative and content related—to the entire class or to just the staff. You are likely to get faster response through Piazza than from direct email to staff. Posting questions or comments to Piazza is optional.

- Problems should be submitted separately following the pset submission instructions, and each problem should have an attached collaboration statement.

STAFF NOTE: Lectures covered: Intro, Proof by Contradiction, WOP

Problem 1.
Show that log\(_7\) n is either an integer or irrational, where n is a positive integer. Use whatever familiar facts about integers and primes you need, but explicitly state such facts.

Solution. The statement to be proved is equivalent to the assertion that, for all positive integers, n, if log\(_7\) n is rational, then it is an integer. This is clearly true for n = 1, since log\(_7\) 1 is the integer zero.

Since log\(_7\) x > 0 for x > 1, if log\(_7\) n not irrational, we would have

\[ \log_7 n = \frac{i}{j} \]  

for some positive integers, i, j.

Now, raising 7 to the power log\(_7\) n, we have from (1)

\[ n = 7^{\log_7 n} = 7^{i/j}. \]

Then, taking \(j\)th powers,

\[ n^j = (7^{i/j})^j = 7^i. \]  

Since \(i, j > 0\), both sides of equation (2) are integers. Also, since the only prime dividing the righthand of (2) is 7, the fact that integers factor uniquely into primes implies that the only prime factor of \(n^j\), and hence the only prime factor of \(n\), is 7. This means that \(n\) can only be a nonnegative power of 7, so log\(_7\) \(n\) must be a nonnegative integer.
Problem 2.
Use the Well Ordering Principle to prove that
\[ n \leq 3^{n/3} \quad (3) \]
for every nonnegative integer, \( n \).

*Hint:* Verify (3) for \( n \leq 4 \) by explicit calculation.

**Solution.** Suppose to the contrary that (3) failed for some nonnegative integer. Then by the WOP, there is a least such nonnegative integer, \( m \).

But \( 0 \leq 3^{0/3} \), so \( m \neq 0 \). Also, \( 1^3 \leq 3^1 \), so taking cube roots, \( 1 \leq 3^{1/3} \), which implies \( m \neq 1 \). Likewise, \( 2^3 \leq 3^2 \), so taking cube roots, \( 2 \leq 3^{2/3} \), which implies \( m \neq 2 \). Similar simple calculations show that \( m \neq 3, 4 \), so we know that \( m \geq 5 \).

Now since \( m > m - 3 \geq 0 \) and \( m \) is the least nonnegative integer for which the inequality (3) fails, the inequality must hold when \( n = m - 3 \). So
\[
3^{m/3} = 3 \cdot 3^{(m-3)/3} \\
\geq 3 \cdot (m - 3) \quad \text{(by (3) for } n = m - 3) \quad (4)
\]

Also,
\[
3 \cdot (m - 3) = 3m - 9 \\
> 3m - 2m \quad \text{since } m > 9/2 \\
= m. \quad (5)
\]

Combining (4) and (5), we get
\[
m \leq 3^{m/3},
\]
contradicting the assumption that (3) fails for \( n = m \).

This contradiction implies that there cannot be a nonnegative integer for which (3) fails. By the WOP, this means that (3) must hold for all nonnegative integers.

Problem 3.
For \( n = 40 \), the value of polynomial \( p(n) := n^2 + n + 41 \) is not prime, as noted in Section 1.1 of the course text. But we could have predicted based on general principles that no nonconstant polynomial can generate only prime numbers.

In particular, let \( q(n) \) be a polynomial with integer coefficients, and let \( c := q(0) \) be the constant term of \( q \).

(a) Verify that \( q(cm) \) is a multiple of \( c \) for all \( m \in \mathbb{Z} \).

**Solution.** Say \( q(n) = c + \sum_{i=1}^{k} a_i n^i \) where \( a_i \in \mathbb{Z} \). Then
\[
q(cm) = c + \sum_{i=1}^{k} a_i (cm)^i = c \left( 1 + \sum_{i=1}^{k} a_i m^i c^{i-1} \right).
\]

(b) Show that if \( q \) is nonconstant and \( c > 1 \), then as \( n \) ranges over the nonnegative integers, \( \mathbb{N} \), there are infinitely many \( q(n) \in \mathbb{Z} \) that are not primes.

*Hint:* You may assume the familiar fact that the magnitude of any nonconstant polynomial, \( q(n) \), grows unboundedly as \( n \) grows.
Solution. If \(|q(cm)| > c > 1\), then \(q(cm)\) won’t be prime because by part (a), it has \(c\) as a factor. Since \(|q(n)|\) grows unboundedly with \(n\), there will be infinitely many different such values of \(q(cm)\) as \(m\) grows.

(c) Conclude immediately that for every nonconstant polynomial, \(q\), there must be an \(n \in \mathbb{N}\) such that \(q(n)\) is not prime.

Solution. By part (b), the only remaining case is when \(c \leq 1\). But in that case \(q(n)\) is not prime for \(n = 0\).