Staff Solutions to Mini-Quiz 5

STAFF NOTE: DAGs, Ch. 9.4; Partial Orders & Equivalence Relations, Ch 9.5–9.10

Problem 1 (4 points).
Let \{A, ..., H\} be a set of tasks that we must complete. The following DAG describes which tasks must be done before others, where there is an arrow from a to b iff a must be done before b.

(a) Write the longest chain.

Solution. A, D, E, G

(b) Write the longest antichain.

Solution. A, B, C

(c) If we allow parallel scheduling, and each task takes 1 minute to complete, what is the minimum amount of time needed to complete all tasks?

Solution. This should be the length of the longest chain, 4 minutes.

Problem 2 (6 points).
Let \(R_1\) and \(R_2\) be two equivalence relations on a set, A. Prove that \(R_1 \cap R_2\) is also an equivalence relation.

Solution. Let \(R \equiv R_1 \cap R_2\).
Proof. We prove that $R$ is an equivalence relation by showing that $R$ is reflexive, symmetric, and transitive.

**Reflexive:** $R_i$ is reflexive because it is an equivalence relation, for $i = 1, 2$. So $(a, a) \in R_i$ for $i = 1, 2$ and all $a \in A$. So, $(a, a) \in (R_1 \cap R_2) = R$ for all $a \in A$, that is, $R$ is reflexive.

**Transitive:** Suppose $(a, b), (b, c) \in R$. Since $R = R_1 \cap R_2$, we have $(a, b), (b, c) \in R_i$ for $i = 1, 2$. But $R_i$ is an equivalence relation, and so is transitive. Therefore, $(a, c) \in R_i$, and so $(a, c) \in R_1 \cap R_2 = R$. This shows that $R$ is transitive.

**Symmetric:** The proof that $R$ is symmetric follows the same format.

Other proofs are possible based on the alternative characterizations of equivalence relations in terms of partitions (Theorem 9.10.4) or having the same functional value (Definition 9.10.2).