Staff Solutions to Mini-Quiz 4

STAFF NOTE: Infinite Cardinality; The Halting Problem, Ch. 7; Number Theory: GCD’s, Ch. 8–8.4; Number Theory: Modular Arithmetic, Ch. 8.5–8.9

Problem 1 (6 points).
Prove that if \( A \) and \( B \) are countable sets, then so is \( A \cup B \).

Solution. Proof. Suppose the list of distinct elements of \( A \) is \( a_0, a_1, \ldots \) and the list of \( B \) is \( b_0, b_1, \ldots \). Then a list of all the elements in \( A \cup B \) is just

\[
a_0, b_0, a_1, b_1, \ldots a_n, b_n, \ldots
\]

(1)

Of course this list will contain duplicates if \( A \) and \( B \) have elements in common, but then deleting all but the first occurrences of each element in list (1) leaves a list of all the distinct elements of \( A \) and \( B \).

STAFF NOTE: If students get stuck, give them the hint that it’s just like the bijection between \( \mathbb{N} \) and \( \mathbb{Z} \) given in the notes (7.2).

Problem 2 (4).
[points = 4] A majority of the following statements are equivalent to each other. List all statements in this majority. Assume that \( n > 0 \) and \( a \) and \( b \) are integers.

1. \( a \equiv b \pmod{n} \)
2. \( a = b \)
3. \( n \mid (a - b) \)
4. \( \exists k \in \mathbb{Z}. a = b + nk \)
5. \( (a - b) \) is a multiple of \( n \)
6. \( \text{rem}(a, n) = \text{rem}(b, n) \)
7. \( a \equiv \text{rem}(b, n) \pmod{n} \)

Solution. 1,3,4,5,6,7