Staff Solutions to Mini-Quiz 3, afternoon

**STAFF NOTE:** Induction, Ch.5–5.3; State Machines: Invariance, Ch. 5.4; Recursive Data & Structural Induction, Ch.6

**Problem 1 (4 points).**
The following state machine describes a procedure that terminates with the product of two nonnegative integers $x$ and $y$ in register $a$. Its states are triples of nonnegative integers $(r, s, a)$. The initial state is $(x, y, 0)$. The transitions are given by the rule that for $s > 0$:

$$(r, s, a) \rightarrow \begin{cases} (2r, s/2, a) & \text{if } s \text{ is even,} \\ (2r, (s-1)/2, a + r) & \text{otherwise.} \end{cases}$$

Circle the predicates below that are preserved invariants for this state machine:

- $[a > 0]$
- $[r < 0]$
- $[r = 0]$
- $[s + a = xy]$
- $[r + a = s]$
- $[rs + a = xy]$

**Solution.** $[rs + a = xy]$, and $[r = 2r + 1]$ are invariants. The second of these is vacuously invariant because it is always false.

**Solution.** $[a > 0]$ is a preserved invariant since $a$ stays the same or increases by $r \geq 0$ at each transition. $[r < 0]$ is a preserved invariant since it is always false. $[r = 0]$ is a preserved invariant since $r$ just gets doubled at each transition. $[rs + a = xy]$ is the preserved invariant that could be used to verify partial correctness. But $[s + a = xy]$ is not preserved, because, for example, $(1, 5, 0) \rightarrow (2, 2, 1)$, and likewise $[r + a = s]$ are preserved because, for example, $(1, 1, 0) \rightarrow (2, 0, 0)$.

**Problem 2 (6 points).**
Prove by induction:

$$\sum_{i=0}^{n} i^3 = \left( \frac{n(n + 1)}{2} \right)^2, \forall n \geq 0$$

(1)

using the equation itself as the induction hypothesis, $P(n)$.
(a) Prove the base case \((n = 0)\).

**Solution.**

\[
\begin{align*}
\frac{(0)(1)^2}{2} &= 0 \\
\end{align*}
\]

Therefore, \(P(0)\) is true. 

(b) Now prove the inductive step.

**Solution.** Let \(P(n) := \left( \sum_{i=0}^{n} i^3 = \frac{n(n+1)^2}{2} \right)\).

Suppose \(P(n)\) is true, we need to show \(P(n + 1)\).

**Proof.**

\[
\begin{align*}
\sum_{i=0}^{n} i^3 &= \left( \frac{n(n+1)}{2} \right)^2 \\
\sum_{i=0}^{n+1} i^3 &= (n + 1)^3 + \left( \frac{n(n+1)}{2} \right)^2 \\
&= (n + 1)^2 \left( \frac{n^2}{2} + (n + 1) \right) \\
&= (n + 1)^2 \left( \frac{n^2 + 4n + 4}{4} \right) \\
&= (n + 1)^2 \left( \frac{(n+2)^2}{4} \right) \\
&= \left( \frac{(n+1)(n+2)}{2} \right)^2 \\
&= P(n + 1)
\end{align*}
\]

Therefore, \(\sum_{i=1}^{n} i^3 = \left( \frac{n(n+1)}{2} \right)^2, \forall n \geq 0\). 