Staff Solutions to Mini-Quiz 2

STAFF NOTE: Predicate Formulas, Ch. 3.6; Sets Relations, Ch 4–4.4; Bijections Finite Cardinality, Ch. 4.5

Problem 1 (4 points).
You’ve seen how certain set identities follow from corresponding propositional equivalences. For example, you proved by a chain of iff’s that

\[(A - B) \cup (A \cap B) = A\]

using the fact that the propositional formula \(P \land \neg Q\) or \(P \land Q\) is equivalent to \(P\).

State a similar propositional equivalence that would justify the key step in a proof for the following set equality organized as a chain of iff’s:

\[A \setminus B = (A - C) \cup (B \cap C) \cup ((A \cup B) \cap C)\]  \hspace{1cm} (1)

(You are not being asked to write out an iff-proof of the equality or to write out a proof of the propositional equivalence. Just state the equivalence.)

Solution. The needed propositional equivalence is that

\[\neg (P \land \neg Q)\]

is equivalent to

\[((\neg P \land \neg (\neg R)) \lor (Q \land R)) \lor ((\neg P \lor Q) \land \neg R)\].  \hspace{1cm} (2)

This problem illustrates the clear correspondence set equalities involving operations, like union and set difference, and corresponding propositional equivalences. The correspondence reduces set equality proofs to proofs of propositional equivalence, allowing for automatic proofs of such set equalities.

Notice that you were not expected to write out a proof of the set equality. But to remind you how the propositional equivalence would be used, we’ve written out the full proof below.

Proof. The equality follows from the fact that

\[x \in \overline{A - B} \text{   iff   } x \in (\overline{A} - C) \cup (B \cap C) \cup ((\overline{A} \cup B) \cap C),\]  \hspace{1cm} (3)

for all \(x\).

To prove (3), let’s define three propositions describing the membership of \(x\) in each of the sets \(A\), \(B\), and \(C\):

\[P \quad ::= \quad x \in A,\]
\[Q \quad ::= \quad x \in B,\]
\[R \quad ::= \quad x \in C.\]
Now, express membership in $A - B$ in terms of $P$, $Q$, and $R$:

\[
x \in A - B \\
\iff \text{ NOT } (x \in A - B) \quad \text{(def of set complement)} \\
\iff \text{ NOT } (x \in (A \cap \overline{B})) \quad \text{(def of } A - B) \\
\iff \text{ NOT } (x \in A \text{ AND } x \in \overline{B}) \quad \text{(def of intersection)} \\
\iff \text{ NOT } (x \in A \text{ AND NOT } (x \in B)) \quad \text{(def of set complement)} \\
\iff \text{ NOT } (P \text{ AND } \overline{Q}) .
\]

Then express membership in

\[
(A - C) \cup (B \cap C) \cup ((A \cup B) \cap \overline{C})
\]

in terms of $P$, $Q$, and $R$:

\[
\text{ OR } x \in (B \cap C) \text{ OR } (x \in (A \cup B) \text{ AND } x \in \overline{C})
\]

\[
\iff \text{ NOT}(P) \text{ AND NOT}(x \in \overline{C}) \text{ OR } ((x \in \overline{A} \text{ OR } Q) \text{ AND NOT}(R))
\]

\[
\iff (\text{ NOT}(P) \text{ AND NOT}(x \in \overline{C})) \text{ OR } ((x \in \overline{A} \text{ OR } Q) \text{ AND NOT}(R))
\]

The membership equivalence (3) now follows from the propositional equivalence (2).

---

**Problem 2 (6 points).**

Let $R : A \rightarrow B$ be a binary relation. Use an arrow counting argument to prove the following generalization of the Mapping Rule 1.

**Lemma.** If $R$ is a function, and $X \subseteq A$, then

\[
|X| \geq |R(X)|.
\]

**Solution.** *Proof.* The proof is virtually a repeat of the arrow-counting proof in the text of Mapping Rule 1, namely:

Since $R$ is a function, at most one arrow leaves each element of $X$, so the number of arrows whose starting point is an element of $X$ is at most the number of elements in $X$. That is,

\[
|X| \geq \#\text{arrows from } X.
\]

Also, each element of $R(X)$ is, by definition, the endpoint of at least one arrow starting from $X$, so there must be at least as many arrows starting from $X$ as the number of elements of $R(X)$. That is,

\[
\#\text{arrows from } X \geq |R(X)|.
\]

Combining these inequalities immediately implies that $|X| \geq |R(X)|$.

An alternative proof appeals to the original Mapping Rule:

*Proof.* Let $R'$ be the relation $R$ restricted to $X$. That is, $R'$ has domain $X$, codomain $R(X)$, and the same arrows as $R$. Then $R'$ is a function because $R$ is, and $R'$ has the $[\geq 1]$ in surjective property by definition of its codomain. Hence the surjective function Mapping Rule 1 applied to the surjective function $R' : X \rightarrow R(X)$ implies that $|X| \geq |R(X)|$. ■
STAFF NOTE: Here’s a repeat of the proof of Mapping Rule 1 to remind students of if need be:

Lemma (Mapping Rule). If $R : A \rightarrow B$ is a surjective function, then

$$|A| \geq |B|.$$ 

Proof. Since $R$ is a function, every element of $A$ contributes at most one arrow to the diagram for $R$, so the number of arrows is at most the number of elements in $A$:

$$|A| \geq \#\text{arrows}.$$ 

Similarly, since $R$ is surjective, every element of $B$ has at least one arrow into it, so there must be at least as many arrows as the number of elements of $B$:

$$\#\text{arrows} \geq |B|.$$ 

Combining these inequalities immediately implies that $|A| \geq |B|$. 
