Staff Solutions to Mini-Quiz 10

Problem 1 (4 points).
What is the coefficient of $x^n$ in the generating function

$$\frac{1 + x}{(1 - x)^2}$$

Solution.

$$(n + 1) + n = 2n + 1$$

Problem 2 (6 points).
A flip of Coin 1 is $x$ times as likely to come up Heads as a flip of Coin 2. A biased random choice of one of these coins made, where the probability of choosing Coin 1 is $w$ times that of Coin 2. The chosen coin is flipped and comes up Heads.

(a) Restate the information above using probabilities and conditional probabilities involving the events

- $C_1$ := Coin 1 was chosen,
- $C_2$ := Coin 2 was chosen,
- $H$ := the chosen coin came up Heads.

Solution. We are given that

$$\Pr[H \mid C_1] = x \Pr[H \mid C_2],$$

and

$$\Pr[C_1] = w \Pr[C_2].$$

(b) State an inequality involving conditional probabilities of the above events that formalizes the assertion “Given that the chosen coin came up Heads, the chosen coin is more likely to have been Coin 1 than Coin 2.”

Solution.

$$\Pr[C_1 \mid H] > \Pr[C_2 \mid H].$$

(c) Prove that, given that the chosen coin came up Heads, the chosen coin is more likely to have been Coin 1 than Coin 2 iff

$$wx > 1.$$
Solution. From (1) and the definition of conditional probability,

\[
\frac{\Pr[H \text{ AND } C_1]}{\Pr[C_1]} = \frac{x \Pr[H \text{ AND } C_2]}{\Pr[C_2]}.
\]

Substituting for \(\Pr[C_1]\), we get

\[
\frac{\Pr[H \text{ AND } C_1]}{w \Pr[C_2]} = \frac{x \Pr[H \text{ AND } C_2]}{\Pr[C_2]},
\]

which implies

\[
\Pr[H \text{ AND } C_1] = w.x \Pr[H \text{ AND } C_2].
\]

Dividing both sides of this inequality by \(\Pr[H]\), yields

\[
\Pr[C_1 \mid H] = w.x \Pr[C_2 \mid H],
\]

so

\[
\Pr[C_1 \mid H] > \Pr[C_2 \mid H] \quad \text{iff} \quad w.x > 1.
\]

\[\blacksquare\]