Staff Solutions to Mini-Quiz 1

Problem 1 (3 points).
Prove that \( \sqrt[3]{35} \) is irrational.

Solution. Proof. Assume for the sake of contradiction that \( \sqrt[3]{35} \) is rational. Under this assumption, there exist integers \( a \) and \( b \) such that

\[
\sqrt[3]{35} = \frac{a}{b},
\]

where \( a \) and \( b \) have no common factor. Now we prove that \( a \) and \( b \) have 5 as a common factor, a contradiction.

\[
\begin{align*}
\sqrt[3]{35} &= \frac{a}{b}, \\
35 &= \frac{a^7}{b^7}, \\
35b^7 &= a^7.
\end{align*}
\]

(1)

Since 5 is a factor of the left-hand side of (1), it is also a factor of the right-hand side, \( a^7 \). By unique factorization, this implies that 5 is a factor of \( a \).

In particular, \( a = 5c \) for some integer \( c \). Thus,

\[
\begin{align*}
35b^7 &= (5c)^7 = 5^7c^7, \\
7b^7 &= 5^6c^7.
\end{align*}
\]

(2)

Since 5 is a factor of the right-hand side of (2), it is also a factor of the left-hand side, \( 7b^7 \). But 5 is not a factor of 7, so by unique factorization it must be a factor of \( b^7 \) and hence of \( b \).

\[
\boxed{\text{Problem 2 (3 points).}}
\]

Prove by the Well Ordering Principle that for all nonnegative integers, \( n \):

\[
\sum_{i=0}^{n} i^3 = \left( \frac{n(n+1)}{2} \right)^2.
\]

Solution. The proof is by contradiction.

Suppose to the contrary that this failed for some \( n \geq 0 \). Then by the WOP, there is a smallest nonnegative integer, \( m \), such this formula does not hold when \( n = m \).

But it clearly holds when \( n = 0 \), which means that \( m \geq 1 \). So \( m-1 \) is nonnegative, and since it is smaller than \( m \), the formula must be true for \( n = m - 1 \). That is,

\[
\sum_{i=0}^{m-1} i^3 = \left( \frac{(m-1)m}{2} \right)^2.
\]

(3)
Now add \( m^3 \) to both sides of equation (3). Then the left hand side equals
\[
\sum_{i=0}^{m} i^3
\]
and the right hand side equals
\[
\left( \frac{(m-1)m}{2} \right)^2 + m^3
\]
Now a little algebra shows that the right hand side equals
\[
\left( \frac{m(m+1)}{2} \right)^2.
\]
That is,
\[
\sum_{i=0}^{m} i^3 = \left( \frac{m(m+1)}{2} \right)^2,
\]
contradicting the fact that our formula does not hold for \( m \).

Problem 3 (4 points).
There are exactly two truth environments (assignments) for the variables \( M, N, P, Q, R, S \) that satisfy the following formula:
\[
(\overline{P} \lor Q) \land (\overline{Q} \lor R) \land (\overline{R} \lor S) \land (\overline{S} \lor P) \land M \land \overline{N}
\]
(a) This could be proved by truth-table. How many rows would the truth table have?

**Solution.** A truth table for a formula with 6 variables has \( 2^6 = 64 \) rows.

(b) Instead of a truth-table, prove this with an argument by cases according to the truth value of \( P \).

**Solution.** Obviously \( M \) must be true and \( N \) must be false. Now we have:

**Case 1** \((P \) is false): In order to have any chance of satisfying clause (4), \( S \) must be false. Similarly, if \( S \) is false, then in order to satisfy clause (3), \( R \) must be false; similarly, \( Q \) must be false.

**Case 2** \((P \) is true): \( Q \) must be true to make clause (1) true and have any chances of making the overall expression true. Similarly, if \( Q \) is true, then \( R \) must be true and if \( R \) is true then \( S \) is true.

Those arguments prove there are at most two satisfying truth environments, but we need to show the two environments we were left with actually satisfy the formula. This can be easily done, by plugging the values into the formula:

If all variables \( P, Q, R, S \) are set to true, then since clause (1) has \( Q \) clause (2) has \( R \), clause (3) has \( S \), and clause (4) has \( P \), then every clause is satisfied, and the full AND-combination is satisfied. If all are false, then since clause (1) has \( \overline{P} \), clause (2) has \( \overline{Q} \), clause (3) has \( \overline{R} \) and clause (4) has \( \overline{S} \), then again every clause is satisfied and the overall proposition is satisfied. So both of those satisfy the proposition.