Staff Solutions to In-Class Problems Week 7, Fri.

STAFF NOTE: Partial Orders & Equivalence Relations, Ch.9.5-9.10

Problem 1.
Show that the set of nonnegative integers partially ordered under the divides relation...

(a) ...has a minimum element.

Solution. 1 is minimum since it divides all nonnegative integers,

(b) ...has a maximum element.

Solution. 0 is maximum because everything divides 0 (including 0 — look up the definition of $| n$).

(c) ...has an infinite chain.

Solution. 1 2 4 8 16 ...is a chain with infinite length.

(d) ...has an infinite antichain.

STAFF NOTE: Hint: The primes.

Solution. The set of prime numbers is infinite. Since no prime divides another, any two primes are incomparable. So the set of prime numbers is an antichain.

(e) What are the minimal elements of divisibility on the integers greater than 1? What are the maximal elements?

Solution. The primes are the minimal elements. There are no maximal elements.

Problem 2.
For each of the binary relations below, state whether it is a strict partial order, a weak partial order, an equivalence relation or none of these. If it is a partial order, state whether it is a path-total order. If it is none, indicate which of the axioms for partial order and equivalence relations it violates.

STAFF NOTE: This problem took longer than expected for students to go through in the class. Parts (f) and (h) are the trickiest parts, usually where students made mistakes. Give hints as needed to get them through faster.

(a) The superset relation, $\supseteq$ on the power set $\text{pow } \{1, 2, 3, 4, 5\}$.
Solution. This is a weak partial order, but not a path-total one. For example, the sets of size 3 form an antichain.

(b) The relation between any two nonnegative integers, $a, b$ that $a \equiv b \pmod{8}$.

Solution. An equivalence relation.

(c) The relation between propositional formulas, $G, H$, that $[G \implies H]$ is valid.

Solution. Violates antisymmetry: $P$ and $\neg(\neg P)$ imply each other but are different expressions. It is transitive, though.

(d) The relation between propositional formulas, $G, H$, that $[G \iff H]$ is valid.

Solution. An equivalence relation.


Solution. Obviously violates transitivity. Irreflexive since nothing beats itself, and antisymmetric.

(f) The empty relation on the set of real numbers.

Solution. It’s vacuously asymmetric and transitive, so it’s a partial order. It’s not path-total. It is not an equivalence relation because it is not reflexive.

(g) The identity relation on the set of integers.

Solution. It’s obviously reflexive, antisymmetric and transitive, so it’s a weak partial order. It’s not path-total. It’s also an equivalence relation since it is symmetric as well.

(h) The divisibility relation on the integers, $\mathbb{Z}$.

Solution. Not antisymmetric: 3 and -3 divide each other. It is transitive and reflexive.

Problem 3.
Let $S$ be a sequence of $n$ different numbers. A subsequence of $S$ is a sequence that can be obtained by deleting elements of $S$.

For example, if

$$S = (6, 4, 7, 9, 1, 2, 5, 3, 8)$$

Then 647 and 7253 are both subsequences of $S$ (for readability, we have dropped the parentheses and commas in sequences, so 647 abbreviates $(6, 4, 7)$, for example).

An increasing subsequence of $S$ is a subsequence of whose successive elements get larger. For example, 1238 is an increasing subsequence of $S$. Decreasing subsequences are defined similarly; 641 is a decreasing subsequence of $S$.

(a) List all the maximum length increasing subsequences of $S$, and all the maximum length decreasing subsequences.
Solution. The maximum length increasing subsequences are 1238 and 1258. The maximum length decreasing subsequences are

641, 642, 643, 653, 753, 953

Now let \( A \) be the set of numbers in \( S \). (So \( A = \{1, 2, 3, \ldots, 9\} \) for the example above.) There are two straightforward ways to path-total order \( A \). The first is to order its elements numerically, that is, to order \( A \) with the \(<\) relation. The second is to order the elements by which comes first in \( S \); call this order \( <_S \). So for the example above, we would have

\[
6 <_S 4 <_S 7 <_S 9 <_S 1 <_S 2 <_S 5 <_S 3 <_S 8
\]

Next, define the partial order \(<\) on \( A \) defined by the rule

\[
a < a' \iff a < a' \text{ and } a <_S a'.
\]

(It’s not hard to prove that \(<\) is strict partial order, but you may assume it.)

(b) Draw a diagram of the partial order, \(<\), on \( A \). What are the maximal elements... the minimal elements?

Solution. The maximal elements are 8 and 9; the minimal are 1, 4, and 6:

(c) Explain the connection between increasing and decreasing subsequences of \( S \), and chains and antichains under \(<\).

Solution. A chain, with its elements listed in numerically increasing order, is an increasing subsequence and an antichain, with its elements listed in numerically decreasing order, is a decreasing subsequence.  


(d) Prove that every sequence, \( S \), of length \( n \) has an increasing subsequence of length greater than \( \sqrt{n} \) or a decreasing subsequence of length at least \( \sqrt{n} \).

Solution. By Dilworth’s Lemma, either a chain or an antichain must have size at least \( \sqrt{n} \), which, by the previous problem part, means there is either an increasing or a decreasing subsequence of this size.

Problem 4.
This problem asks for a proof of Lemma 9.6.2 showing that every weak partial order can be represented by (is isomorphic to) a collection of sets partially ordered under set inclusion (\( \subseteq \)). Namely,

Lemma. Let \( \leq \) be a weak partial order on a set, \( A \). For any element \( a \in A \), let

\[
\begin{align*}
L(a) &:= \{ b \in A \mid b \leq a \}, \\
\mathcal{L} &:= \{ L(a) \mid a \in A \}.
\end{align*}
\]

Then the function \( L() : A \to \mathcal{L} \) is an isomorphism from the \( \leq \) relation on \( A \), to the subset relation on \( \mathcal{L} \).

(a) Prove that the function \( L() : A \to \mathcal{L} \) is a bijection.

Solution. By definition, \( L() \) is a surjective function onto \( \mathcal{L} \), so all we have to do is prove it is an injection. To prove this, suppose \( L((a)) = L(b) \). Now since \( a \in L((a)) \) by reflexivity, we also have \( a \in L(b) \). This means \( a \leq b \). Likewise, \( b \leq a \). Hence \( a = b \), by antisymmetry.

(b) Complete the proof by showing that

\[
a \leq b \quad \text{iff} \quad L(a) \subseteq L(b)
\]

for all \( a, b \in A \).

Solution. For the left-to-right direction, suppose \( a \leq b \). To prove that \( L(a) \subseteq L(b) \), suppose \( c \in L(a) \), which means that \( c \leq a \). So by transitivity, \( c \leq b \), which means \( c \in L(b) \). Hence every \( c \in L(a) \) is also in \( L(b) \), which proves containment.

For the right-to-left direction, suppose \( L(a) \subseteq L(b) \). But \( a \in L(a) \) by reflexivity, so \( a \in L(b) \), which means that \( a \leq b \).

Problem 5.
The equivalence classes of an equivalence relation form a partition of the domain.

Namely, let \( R \) be an equivalence relation on a set, \( A \), and define the equivalence class of an element \( a \in A \) to be

\[
[a]_R := \{ b \in A \mid a \ R b \}.
\]

That is, \( [a]_R = R(a) \).

(a) Prove that every block is nonempty and every element of \( A \) is in some block.

Solution. Proof. Since \( R \) is reflexive, any element \( a \in A \) is a member of the block \([a]_R \) and also each block \([a]_R \) is nonempty.

(b) Prove that if \([a]_R \cap [b]_R \neq \emptyset \), then \( a \ R b \). Conclude that the sets \([a]_R \) for \( a \in A \) are a partition of \( A \).
Solution. Proof. Suppose $c \in [a]_R \cap [b]_R$, that is, $a R c$ and $b R c$. Since $b R c$, we have $c R b$ by symmetry. Since $a R c$, we now conclude by transitivity that $a R b$. 

This and part (a) mean that the sets $[a]_R$ partition $A$, by the definition of partition.

(c) Prove that $a R b$ iff $[a]_R = [b]_R$.

Solution. Proof. (left to right): Suppose $a R b$. Now $x \in [b]_R$ means $b R x$, and since $a R b$ by part (b), we again invoke transitivity to conclude that $a R x$. That is, $x \in [a]_R$. So every element of $[b]_R$ is in $[a]_R$, which means that $[b]_R \subseteq [a]_R$. By symmetry, $[a]_R \subseteq [b]_R$, so in fact $[a]_R = [b]_R$.

(right to left): Suppose $[a]_R = [b]_R$, then since $b \in [b]_R$, we have $b \in [a]_R$, so $a R b$ by definition of $[a]_R$. 
