Staff Solutions to In-Class Problems Week 4, Mon.

Problem 1.  
Prove by induction:

\[ 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}, \]

for all \( n > 1 \).

Solution.  Proof. (By Induction). The induction hypothesis, \( P(n) \), is the inequality (1).

Base Case \((n = 2)\): The LHS of (1) in this case is \( 1 + 1/4 \) and the RHS is \( 2 - 1/2 \), and

\[ \text{LHS} = 5/4 < 6/4 = 3/2 = \text{RHS}, \]

so inequality (1) holds, and \( P(2) \) is proved.

Inductive Step: Let \( n \geq 2 \) be a nonnegative integer, and assume \( P(n) \) in order to prove \( P(n + 1) \). That is, we assume (1). Adding \( 1/(n + 1)^2 \) to both sides of this inequality yields

\[
\begin{align*}
1 + \frac{1}{4} + \cdots + \frac{1}{n^2} + \frac{1}{n(n+1)^2} &< 2 - \left( \frac{1}{n} + \frac{1}{(n+1)^2} \right) \\
&= 2 - \left( \left( \frac{1}{n} - \frac{1}{(n+1)^2} \right) \right) \\
&= 2 - \frac{n^2 + 2n + 1 - n}{n(n+1)^2} \\
&= 2 - \frac{n^2 + n}{n(n+1)^2} - \frac{1}{n(n+1)^2} \\
&= 2 - \frac{1}{n + 1} - \frac{1}{n(n+1)^2} \\
&< 2 - \frac{1}{n + 1} \quad \text{(since } n > 0 \text{).}
\end{align*}
\]

So we have proved \( P(n + 1) \).

This completes the induction case, and we conclude by Induction that \( \forall n \geq 2, P(n) \).

Problem 2. (a)  
Prove by induction that a \( 2^n \times 2^n \) courtyard with a \( 1 \times 1 \) statue of Bill in a corner can be covered with L-shaped tiles. (Do not assume or reprove the (stronger) result of Theorem 5.1.2 that Bill can be placed anywhere. The point of this problem is to show a different induction hypothesis that works.)
**Solution.** Let \( P(n) \) be the proposition Bill can be placed in a corner of a \( 2^n \times 2^n \) courtyard with a proper tiling of the remainder with L-shaped tiles.

**Base case:** \( P(0) \) is true because Bill fills the whole courtyard.

**Inductive step:** Assume that \( P(n) \) is true for some \( n \geq 0 \); that is, there exists a tiling of the \( 2^n \times 2^n \) courtyard leaving Bill in a corner.

To prove, \( P(n+1) \), divide the \( 2^{n+1} \times 2^{n+1} \) courtyard into four quadrants, each \( 2^n \times 2^n \). One quadrant will contain the corner designated for Bill. By induction hypothesis, we can get Bill into some corner of the quadrant, which means we can actually get him into *any* desired corner of the quadrant by rotating the tiling of the quadrant. So place Bill in the designated corner of the quadrant, and tile the rest of the quadrant.

Now tile the remaining three quadrants, leaving a tile space open in the quadrant corners that are in the middle of the whole \( 2^{n+1} \times 2^{n+1} \) courtyard (as in the diagram in the proof of Theorem 5.1.2). These three spaces form an L-shape that can be filled with a single L-shaped tile, completing the full courtyard tiling. This proves \( P(n+1) \), completing the proof by induction that a square courtyard with side length any power of 2 can be tiled with Bill in a corner.

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**Problem 3.**

Any amount of 12 or more cents postage can be made that using only 3¢ and 7¢ stamps. Prove this by induction using the induction hypothesis

\[
S(n) ::= n + 12 \text{ cents postage can be made using only } 3¢ \text{ and } 7¢ \text{ stamps.}
\]

**Solution.**

**Proof.** **Base case** \((n = 0): 12¢ \text{ postage can be made with four } 3¢ \text{ stamps.}**

**Inductive step:** We assume the strong hypothesis that \( S(k) \) for \( n \geq k \geq 0 \). Now we must prove \( S(n+1) \).

The proof is by cases:

- **Case** \( n = 0 \): \( S(0+1) \) holds because \( 12+1=13 \text{ cents postage can be made using one } 7¢ \text{ and two } 3¢ \text{ stamps.} \)
- **Case** \( n = 1 \): \( S(1+1) \) holds because \( (1+1) + 12 = 14¢ \text{ postage can be made using two } 7¢ \text{ stamps.} \)
- **Case** \( n \geq 2 \): Since \( n > n - 2 \geq 0 \), we know by strong induction that \( S(n-2) \) holds. But including an extra \( 3¢ \) stamp in the collection of \( 3¢ \) and \( 7¢ \) stamps that made \( (n-2) + 12 \text{ cents gives a collection that makes } (n-2) + 12 + 3 = (n+1) + 12 \text{ cents, which proves } S(n+1). \)

Since \( S(n+1) \) holds in any case, the inductive step has been proved.

It follows by strong induction \( S(n) \) holds for all \( n \in \mathbb{N} \). That is, every amount of postage of 12 cents or more can be made with \( 3¢ \) and \( 7¢ \) stamps.

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**Problem 4.**

Find the flaw in the following bogus proof that \( a^n = 1 \) for all nonnegative integers \( n \), whenever \( a \) is a nonzero real number.
Bogus proof. The proof is by induction on $n$, with hypothesis

$$P(n) ::= \forall k \leq n, a^k = 1,$$

where $k$ is a nonnegative integer valued variable.

**Base Case:** $P(0)$ is equivalent to $a^0 = 1$, which is true by definition of $a^0$. (By convention, this holds even if $a = 0$.)

**Inductive Step:** By induction hypothesis, $a^k = 1$ for all $k \in \mathbb{N}$ such that $k \leq n$. But then

$$a^{n+1} = a^n \cdot a^n = \frac{1 \cdot 1}{1} = 1,$$

which implies that $P(n + 1)$ holds. It follows by induction that $P(n)$ holds for all $n \in \mathbb{N}$, and in particular, $a^n = 1$ holds for all $n \in \mathbb{N}$.

Solution. The flaw comes in the inductive step, where we implicitly assume $n \geq 1$ in order to talk about $a^{n-1}$ in the denominator (otherwise the exponent is not a nonnegative integer, so we cannot apply the inductive hypothesis). We checked the base case only for $n = 0$, so we are not justified in assuming that $n \geq 1$ when we try to prove the statement for $n + 1$ in the inductive step. And of course the proposition first breaks precisely at $n = 1$. 

\[\square\]