Staff Solutions to In-Class Problems Week 3, Wed.

Problem 1.  
Set Formulas and Propositional Formulas.  
(a) Verify that the propositional formula \((P \text{ AND } Q) \text{ OR } (P \text{ AND } \neg Q)\) is equivalent to \(P\).

**STAFF NOTE:** If students use truth tables, suggest they try again using cases and/or algebra.

**Solution.** There is a simple verification by truth table with 4 rows which we omit.

There is also a simple cases argument: if \(Q\) is \(T\), then the formula simplifies to \((P \text{ AND } F) \text{ OR } (P \text{ AND } T)\) which further simplifies to \((F \text{ OR } P)\) which is equivalent to \(P\).

Otherwise, if \(Q\) is \(F\), then the formula simplifies to \((P \text{ AND } T) \text{ OR } (P \text{ AND } F)\) which is likewise equivalent to \(P\).

Finally, there is a proof by propositional algebra:

\[
(P \text{ AND } Q) \text{ OR } (P \text{ AND } \neg Q) \iff P \text{ AND } Q \text{ OR } Q \iff P \text{ AND } T \iff P.
\]

(b) Prove that \(A = (A - B) \cup (A \cap B)\) for all sets, \(A, B\), by showing

\[
x \in A \iff x \in (A - B) \cup (A \cap B)
\]

for all elements, \(x\), using the equivalence of part (a) in a chain of \(\iff\)'s.

**Solution.** Two sets are equal iff they have the same elements, that is, \(x\) is in one set iff \(x\) is in the other set, for any \(x\). We’ll now prove this for \(A\) and \((A - B) \cup (A \cap B)\).

\[
x \in (A - B) \cup (A \cap B) \\
\iff x \in (A - B) \text{ OR } x \in (A \cap B) \quad \text{(by def of } \cup) \\
\iff (x \in A \text{ AND } \neg x \in B) \text{ OR } (x \in A \text{ AND } x \in B) \quad \text{(by def of } \cap \text{ and } -) \\
\iff (P \text{ AND } \neg Q) \text{ OR } (P \text{ AND } Q) \quad \text{(where } P := [x \in A] \text{ and } Q := [x \in B]) \\
\iff P \quad \text{(by part (a))} \\
\iff x \in A \quad \text{(by def of } P)\]

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1The set difference, \(A - B\), of sets \(A\) and \(B\) is

\[
A - B := \{a \in A \mid a \notin B\}.
\]
STAFF NOTE: Ask your students if they can now see how a computer could automatically check such equalities between set formulas involving the basic set operators like $\cup, \cap, -, \ldots$? The answer is that proving such equalities reduces to verifying equivalence of corresponding propositional formulas as above.

Problem 2.
Subset take-away\(^2\) is a two player game involving a fixed finite set, $A$. Players alternately choose nonempty subsets of $A$ with the conditions that a player may not choose

- the whole set $A$, or
- any set containing a set that was named earlier.

The first player who is unable to move loses the game.

For example, if $A$ is $\{1\}$, then there are no legal moves and the second player wins. If $A$ is $\{1, 2\}$, then the only legal moves are $\{1\}$ and $\{2\}$. Each is a good reply to the other, and so once again the second player wins.

The first interesting case is when $A$ has three elements. This time, if the first player picks a subset with one element, the second player picks the subset with the other two elements. If the first player picks a subset with two elements, the second player picks the subset whose sole member is the third element. Both cases produce positions equivalent to the starting position when $A$ has two elements, and thus leads to a win for the second player.

Verify that when $A$ has four elements, the second player still has a winning strategy.\(^3\)

STAFF NOTE: Suggest that students break up into opposing teams and play a few games to be sure they understand the rules — and get an idea for a winning strategy.

Solution. There are way too many cases to work out by hand if we tried to list all possible games. But the elements of $A$ all behave the same, so we can cut to a small number of cases using the fact that permuting around the elements of $A$ in any game yields another possible game. We can do this by not mentioning specific elements of $A$, but instead using the variables $a, b, c, d$ whose values will be the four elements of $A$.

We consider two cases for the move of the Player 1 when the game starts:

1. Player 1 chooses a one element or a three element subset. Then Player 2 should choose the complement of Player one’s choice. The game then becomes the same as playing the $n = 3$ game on the three element set chosen in this first round, where we know Player 2 has a winning strategy.

2. Player 1 chooses a subset of 2 elements. Let $a, b$ be these elements, that is, the first move is $\{a, b\}$. Player 2 should choose the complement, $\{c, d\}$, of Player 1’s choice. We then have the following subcases:

   (a) Player 1’s second move is a one element subset, $\{a\}$. Player 2 should choose $\{b\}$. The game is then reduced to the two element game on $\{c, d\}$ where Player 2 has a winning strategy.

   (b) Player 1’s second move is a two element subset, $\{a, c\}$. Player 2 should choose its complement, $\{b, d\}$. This leads to two subsubcases:

   i. Player 1’s third move is one of the remaining sets of size two, $\{a, d\}$. Player 2 should choose its complement, $\{b, c\}$. The remaining possible moves are the four sets of size 1, where the Player 2 clearly wins after two more rounds.

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\(^3\)David Gale worked out some of the properties of this game and conjectured that the second player wins the game for any set $A$. This remains an open problem.
ii. Player 1’s third move is a one element set, \{a\}. Player 2 should choose \{b\}. The game is then reduced to the two element game on \{c, d\} where Player 2 has a winning strategy.

So in all cases, Player 2 has a winning strategy in the Gale game for \(n = 4\).

**Problem 3.**
Forming a pair \((a, b)\) of items \(a\) and \(b\) is a mathematical operation that we can safely take for granted. But when we’re trying to show how all of mathematics can be reduced to set theory, we need a way to represent the pair \((a, b)\) as a set.

(a) Explain why representing \((a, b)\) by \(\{a, b\}\) won’t work.

**Solution.** The order of the elements gets lost: \((a, b)\) and \((b, a)\) would have the same representation.

(b) Explain why representing \((a, b)\) by \(\{a, \{b\}\}\) won’t work either. *Hint:* What pair does \(\{\{1\}, \{2\}\}\) represent?

**Solution.** It could equally well represent the pairs \((\{2\}, 1)\) and \((\{1\}, 2)\), so the pair being “represented” can’t be uniquely determined.\(^4\)

(c) Define

\[
\text{pair}(a, b) := \{a, \{a, b\}\}.
\]

Explain why representing \((a, b)\) as \(\text{pair}(a, b)\) uniquely determines \(a\) and \(b\). *Hint:* Sets can’t be indirect members of themselves: \(a \in a\) never holds for any set \(a\), and neither can \(a \in b \in a\) hold for any \(b\).

**Solution.** Notice that \(\{a, b\} \notin a\) because otherwise \(a\) would indirectly be a member of itself, namely \(a \in \{a, b\} \in a\), which sets don’t do.\(^5\). So of the two elements in \(\text{pair}(a, b)\), \(a\) must be the element that is a member of the other one. If there are two elements in this other set, then \(b\) is the element that is not equal to \(a\), otherwise \(b\) must equal \(a\).

**STAFF NOTE:** Tell students not to worry if they don’t get through the next two problems. The first one about finite ordinals gives more practice with the membership and containment relations, but we won’t make further use of the ordinals. The second is more practice with predicate formulas than learning about sets.

**Problem 4.**
For any set \(x\), define \(\text{next}(x)\) to be the set consisting of all the elements of \(x\), along with \(x\) itself:

\[
\text{next}(x) := x \cup \{x\}
\]

Now we can define a sequence of sets \(v_0, v_1, v_2, \ldots\) called the *finite ordinals* with a simple recursive recipe:

\[
v_0 := \emptyset,
\]

\[
v_{n+1} := \text{next}(v_n).
\]

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\(^4\)Thanks to Nursen Ogutveren and his team, Spring ’13.

\(^5\)By the Foundation Axiom, Section 7.3.2
So we have,

\[ v_1 := \{\emptyset\} \]
\[ v_2 := \{\emptyset, \emptyset\} \]
\[ v_3 := \{\emptyset, \emptyset, \emptyset\} \]

The finite ordinals are kind of weird, but have some engaging properties, and more important, they turn out to play a significant role in set theory.

(a) Prove that

\[ v_{n+1} = \{v_0, v_1, \ldots, v_n\} \]. \hspace{1cm} (1)

**Solution. Proof.** Proof by contradiction using WOP.

Suppose equation (1) fails for some nonnegative integer, \( n \). The by WOP, there is a least integer \( m \) for which it fails.

Now (1) holds for \( n = 0 \) since

\[ v_{0+1} = v_1 := \text{next}(v_0) := v_0 \cup \{v_0\} = \emptyset \cup \{v_0\} = \{v_0\} \].

So \( m \geq 1 \).

Since \( m \) is minimal and \( m - 1 \geq 0 \), equation (1) must hold for \( m - 1 \), namely

\[ v_m = v_{(m-1)+1} = \{v_0, v_1, \ldots, v_{m-1}\} \]. \hspace{1cm} (2)

But then

\[
\begin{align*}
v_{m+1} &:= \text{next}(v_m) \\
&:= v_m \cup \{v_m\} \\
&= \{v_0, v_1, \ldots, v_{m-1}\} \cup \{v_m\} \hspace{1cm} \text{(by (2))} \\
&= \{v_0, v_1, \ldots, v_{m-1}, v_m\}.
\end{align*}
\]

So in fact \( m \) also satisfies (1), a contradiction. Hence, equation (1) must hold for all \( n \in \mathbb{N} \).

(b) Conclude that \( |v_n| = n \).

*Hint:* A set cannot be a member of itself.\(^6\)

**STAFF NOTE:** You can’t claim this just from equation (1): you need to be sure that all the elements are different.

(c) Conclude that if \( \mu, \nu, \rho \) are finite ordinals and \( \mu \in \nu \in \rho \), then \( \mu \in \rho \). Likewise, if \( \mu, \nu \) are different finite ordinals, then \( \nu \in \mu \) OR \( \mu \in \nu \).

**Solution.** We need to show that \( m < n \) implies \( v_m \neq v_n \). But this follows because if \( m < n \), then \( v_m \in v_n \) by (1). Now if \( v_m \) was equal to \( v_n \) it would be a member of itself, which sets don’t do..

(b) Conclude that if \( \mu, \nu, \rho \) are finite ordinals and \( \mu \in \nu \in \rho \), then \( \mu \in \rho \). Likewise, if \( \mu, \nu \) are different finite ordinals, then \( \nu \in \mu \) OR \( \mu \in \nu \).

**Solution.** Now from equation (1) we have the \( v_m \in v_n \) iff \( m < n \). So \( \mu \in v \in \rho \) is equivalent to

\[ \mu = v_m, \nu = v_r, \rho = v_r \]

for some integers \( m < n < r \). But then \( m < r \) so \( \mu = v_m \in v_r = \rho \). That is, \( \mu \in \rho \) as required.

Likewise, if \( \mu \) and \( \nu \) are distinct ordinals, then \( \mu = v_m \) and \( \nu = v_n \) for some nonnegative integers \( m \neq n \). But then either \( m < n \) or \( n < m \), in which case \( \mu \in \nu \) or \( \nu \in \mu \), respectively.

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\(^6\)By the Foundation Axiom, Section 7.3.2.
Problem 5.

A formula of set theory is a predicate formula that only uses the predicate \( x \in y \). The domain of discourse is the collection of sets, and \( x \in y \) is interpreted to mean that \( x \) and \( y \) are variables that range over sets, and \( x \) is one of the elements in \( y \).

For example, since \( x \) and \( y \) are the same set iff they have the same members, here's how we can express equality of \( x \) and \( y \) with a formula of set theory:

\[
(x = y) ::= \forall z. (z \in x \iff z \in y).
\] (3)

Express each of the following assertions about sets by a formula of set theory.

(a) \( x = \emptyset \).

**Solution.** \( \forall z. \text{NOT}(z \in x). \)

(b) \( x = \{y, z\} \).

**Solution.** \( \forall w. w \in x \iff (w = y \text{ OR } w = z). \)

(c) \( x \subseteq y \). (\( x \) is a subset of \( y \) that might equal \( y \).)

**Solution.** \( \forall z. z \in x \implies z \in y. \)

Now we can explain how to express \( x \) is a proper subset of \( y \) as a set theory formula using things we already know how to express. Namely, letting \( x \neq y \) abbreviate \( \text{NOT}(x = y) \), the expression

\[
(x \subseteq y \text{ AND } x \neq y),
\]

describes a formula of set theory that means \( x \subset y \).

From here on, feel free to use any previously expressed property in describing formulas for the following:

(d) \( x = y \cup z \).

**Solution.** \( \forall w. w \in x \iff (w \in y \text{ OR } w \in z). \)

(e) \( x = y - z \).

**Solution.** \( \forall w. w \in x \iff (w \in y \text{ AND } \text{NOT}(w \in z)). \)

**STAFF NOTE:** Now other ways to express \( x \subset y \) are

\[
\exists z. (y = x \cup z \text{ AND } z \neq \emptyset).
\]

and

\[
(x - y) = \emptyset.
\]

(f) \( x = \text{pow}(y) \).

**Solution.** \( \forall z. z \in x \iff z \subseteq y. \)
(g) \( x = \bigcup_{z \in y} z \). This means that \( y \) is supposed to be a collection of sets, and \( x \) is the union of all them. A more concise notation for \( \bigcup_{z \in y} z \) is simply \( \bigcup y \).

**Solution.** \( \forall w. w \in x \iff \exists z. (z \in y \text{ AND } w \in z) \).