Staff Solutions to In-Class Problems Week 2, Mon.

Problem 1.
Prove that if \( a \cdot b = n \), then \( a \) or \( b \) must be \( \leq \sqrt{n} \), where \( a, b, \) and \( n \) are nonnegative integers. \textit{Hint:} by contradiction, Section 1.8.

\textbf{Solution.} \textit{Proof.} Suppose to the contrary that \( a > \sqrt{n} \) and \( b > \sqrt{n} \). Then
\[
 a \cdot b > \sqrt{n} \cdot \sqrt{n} = n,
\]
contradicting the fact that \( a \cdot b = n \). \hfill \blacksquare

Problem 2.
Generalize the proof of Theorem 1.8.1 repeated below that \( \sqrt{2} \) is irrational. For example, how about \( \sqrt{3} \) ? Remember that an irrational number is a number that cannot be expressed as a ratio of two integers.

\textbf{Theorem.} \( \sqrt{2} \) is an irrational number.

\textit{Proof.} The proof is by contradiction: assume that \( \sqrt{2} \) is rational, that is,
\[
 \sqrt{2} = \frac{n}{d},
\]
where \( n \) and \( d \) are integers. Now consider the smallest such positive integer denominator, \( d \).
We will prove in a moment that the numerator, \( n \), and the denominator, \( d \), are both even. This implies that
\[
\frac{n}{2} \quad \frac{d}{2}
\]
is a fraction equal to \( \sqrt{2} \) with a smaller positive integer denominator, a contradiction.

\textit{Since the assumption that} \( \sqrt{2} \) \textit{is rational leads to this contradiction, the assumption must be false. That is,} \( \sqrt{2} \) \textit{is indeed irrational.} This italicized comment on the implication of the contradiction normally goes without saying, but since this is an early example of proof by contradiction, we’ve said it.

To prove that \( n \) and \( d \) have 2 as a common factor, we start by squaring both sides of \((1)\) and get
\[
2 = n^2 / d^2,
\]
so
\[
2d^2 = n^2. \tag{2}
\]
So 2 is a factor of \( n^2 \), which is only possible if 2 is in fact a factor of \( n \).
This means that \( n = 2k \) for some integer, \( k \), so
\[
 n^2 = (2k)^2 = 4k^2. \tag{3}
\]
Combining (2) and (3) gives $2d^2 = 4k^2$, so
$$d^2 = 2k^2.$$  \hfill (4)

So 2 is a factor of $d^2$, which again is only possible if 2 is in fact also a factor of $d$, as claimed.

**Solution.** We prove that for any $n > 1$, $\sqrt[n]{2}$ is irrational.

**STAFF NOTE:** A similar but somewhat more interesting generalization to suggest to students who finish quickly is $\sqrt[n]{3}$ is irrational for $n > 1$. The proof is the same as above with “2” replaced by “3,” except that now the needed claim is that if $a^n$ is divisible by 3, then so is $a$, which requires appealing to prime factorization as in the previous paragraph. See the comments at the end of the solution.

**Proof.** The proof is by contradiction.

Assume to the contrary that $\sqrt[n]{2}$ is rational. Under this assumption, there exist integers $a$ and $b$ with $\sqrt[n]{2} = a/b$, where $b$ is the smallest such positive integer denominator. Now we prove that $a$ and $b$ are both even, so that
$$\frac{a}{2} \quad \frac{b}{2}$$
is a fraction equal to $\sqrt[n]{2}$ with a smaller positive integer denominator, a contradiction.

$$\sqrt[n]{2} = \frac{a}{b}$$
$$2 = \frac{a^n}{b^n}$$
$$2b^n = a^n.$$  \hfill (5)

The lefthand side of the last equation is even, so $a^n$ is even. This implies that $a$ is even as well (see below for justification).

In particular, $a = 2c$ for some integer $c$. Thus,
$$2b^n = (2c)^n = 2^n c^n,$$
$$b^n = 2^{n-1} c^n.$$

Since $n - 1 > 0$, the righthand side of the last equation is an even number, so $b^n$ is even. But this implies that $b$ must be even as well, contradicting the fact that $a/b$ is in lowest terms. \hfill □

Now we justify the claim that if $a^n$ is even, so is $a$.

There is a simple proof by contradiction: suppose to the contrary that $a$ is odd. It’s a familiar (and easily verified\(^1\)) fact that the product of two odd numbers is odd, from which it follows that the product of any finite number of odd numbers is odd, so $a^n$ would also be odd, contradicting the fact that $a^n$ is even.

More generally for any integers $m, k > 0$, if $m^k$ is divisible by a prime number, $p$, then $m$ must be divisible by $p$. This follows from the unique factorization of an integer into primes (see Section 8.4): the primes in the factorization of $m^k$ are precisely the primes in the factorization of $m$ repeated $k$ times. Using this fact, the proof above carries over to prove a broader generalization:

**Theorem.** For all positive integers $n$ and $k$,

$\sqrt[n]{m}$ must be either an integer or an irrational.

\(^1\)Two odd integers can be written as $2x + 1$ and $2y + 1$ for some integers $x$ and $y$. Then their product is also odd because it equals $2z + 1$ where $z = 2(xy + x + y) + 1$. 

1 Two odd integers can be written as $2x + 1$ and $2y + 1$ for some integers $x$ and $y$. Then their product is also odd because it equals $2z + 1$ where $z = 2(xy + x + y) + 1$. 

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For an even broader generalization, see Problem 1.11.

Problem 3.
If we raise an irrational number to an irrational power, can the result be rational? Show that it can by considering $\sqrt{2}^\sqrt{2}$ and arguing by cases.

Solution. We want to find irrational numbers $a, b$ such that $a^b$ is rational. We argue by cases.

Case 1: $[\sqrt{2}^\sqrt{2}$ is rational]. Let $a = b = \sqrt{2}$. Now $a$ and $b$ are irrational, since $\sqrt{2}$ is irrational as we know. Also, $a^b$ is rational by case hypothesis. So we have found the required $a$ and $b$ in this case.

Case 2: $[\sqrt{2}^\sqrt{2}$ is irrational]. Let $a = \sqrt{2}^\sqrt{2}$ and $b = \sqrt{2}$. Then $a$ is irrational by case hypothesis, we know $b$ is irrational, and

$$a^b = \left(\sqrt{2}^\sqrt{2}\right)^\sqrt{2} = \sqrt{2}^\sqrt{2}\cdot\sqrt{2} = \sqrt{2}^2 = 2,$$

which is rational. So we have found the required $a$ and $b$ in this case also.

So in any case, there will be irrational $a, b$ such that $a^b$ is rational. Note that we have no clue about which case is true, but that didn’t matter.

Problem 4.
The fact that there are irrational numbers $a, b$ such that $a^b$ is rational was proved in Problem 1.5 of the course text. Unfortunately, that proof was nonconstructive: it didn’t reveal a specific pair, $a, b$, with this property. But in fact, it’s easy to do this: let $a := \sqrt{2}$ and $b := 2 \log_2 3$.

We know $\sqrt{2}$ is irrational, and obviously $a^b = 3$. Finish the proof that these values for $a, b$ work, by showing that $2 \log_2 3$ is irrational.

Solution. Proof. Suppose to the contrary that $2 \log_2 3$ was rational. Then $\log_2 3$ must also be rational, say

$$\log_2 3 = m/n$$

for some integers $m, n > 0$. So $m = n \log_2 3$. Now raising 2 to each side of this equation gives

$$2^m = 2^n \log_2 3 = (2^{\log_2 3})^n = 3^n. \quad (5)$$

But this is impossible, since right hand side of (5) is divisible by 3 (because $n > 0$), and the left hand side is not.

So $2 \log_2 3$ must be irrational.