Staff Solutions to In-Class Problems Week 14, Wed.

STAFF NOTE: Infinite Expectation, Random Walk, Ch. 19.8, 20.1

Problem 1.
A gambler is placing $1 bets on the “1st dozen” in roulette. This bet wins when a number from one to twelve comes in, and then the gambler gets his $1 back plus $2 more. Recall that there are 38 numbers on the roulette wheel.

The gambler’s initial stake in $n$ and his target is $T$. He will keep betting until he runs out of money (“goes broke”) or reaches his target. Let $w_n$ be the probability of the gambler winning, that is, reaching target $T$ before going broke.

(a) Write a linear recurrence for $w_n$; you need not solve the recurrence.

Solution. The probability of winning a bet is $\frac{12}{38}$. Thus, by the Law of Total Probability 17.5.1,

$$w_n = \Pr[\text{win with } n \text{ start } | \text{ won 1st bet}] \cdot \Pr[\text{won 1st bet}]$$

$$+ \Pr[\text{win with } n \text{ start } | \text{ lost 1st bet}] \cdot \Pr[\text{lost 1st bet}] $$

$$= \Pr[\text{win with } n + 2 \text{ start}] \cdot \Pr[\text{won 1st bet}]$$

$$+ \Pr[\text{win with } n - 1 \text{ start}] \cdot \Pr[\text{lost 1st bet}],$$

so

$$w_n = \frac{12}{38} w_{n+2} + \frac{26}{38} w_{n-1}. $$

Letting $m := n + 2$ we get

$$w_m = \frac{38}{12} w_{m-2} - \frac{26}{12} w_{m-3}. $$

As boundary conditions, we have

$$w_0 = 0, w_T = 1.$$ 

(b) Let $e_n$ be the expected number of bets until the game ends. Write a linear recurrence for $e_n$; you need not solve the recurrence.

Solution. By the Law of Total Expectation, Theorem 18.4.5,

$$e_n = (1 + \text{Ex[#bets with } n \text{ start } | \text{ won 1st bet}]) \cdot \Pr[\text{won 1st bet}]$$

$$+ (1 + \text{Ex[#bets with } n \text{ start } | \text{ lost 1st bet}]) \cdot \Pr[\text{lost 1st bet}]$$

$$= (1 + \text{Ex[#bets with } n + 2 \text{ start}]) \cdot \Pr[\text{won 1st bet}]$$

$$+ (1 + \text{Ex[number of bets starting with } n - 1]) \cdot \Pr[\text{lost 1st bet}],$$

so

$$e_n = (e_{n+2} + 1) \frac{12}{38} + (1 + e_{n-1}) \frac{26}{38}.$$
Letting \( m := n + 2 \) we get
\[
e_m = \frac{38}{12} e_{m-2} - \frac{26}{12} e_{m-3} - \frac{38}{12}
\]
As boundary conditions, we have
\[
e_0 = e_T = 0.
\]

**Problem 2.**
In a gambler’s ruin scenario, the gambler makes independent $1 bets, where the probability of winning a bet is \( p \) and of losing is \( q := 1 - p \). The gambler keeps betting until he goes broke or reaches a target of \( T \) dollars.

Suppose \( T = \infty \), that is, the gambler keeps playing until he goes broke. Let \( r \) be the probability that starting with \( n > 0 \) dollars, the gambler’s stake ever gets reduced to \( n - 1 \) dollars.

(a) Explain why
\[
r = q + pr^2.
\]

**Solution.** By Total Probability
\[
r = \Pr[\text{ever down $1$ | lose the first bet}] \Pr[\text{lose the first bet}]
+ \Pr[\text{ever down $1$ | win the first bet}] \Pr[\text{win the first bet}]
= q + p \Pr[\text{ever down $1$ | win the first bet}]
\]
But
\[
\Pr[\text{ever down $1$ | win the first bet}]
= \Pr[\text{ever down $2$}]
= \Pr[\text{down another $1$ | already down $1$}] \cdot \Pr[\text{ever down $1$}]
= r \cdot r = r^2.
\]
Note that the problem assumes the more or less obvious fact that \( r \) does not depend on \( n > 0 \)—which would not be true if \( T \) was finite.

A rigorous explanation for this is that, since there is no finite upper cutoff \( T \), any path starting at \( n \) and reaching \( n - 1 \) for the first time could be shifted up or down by \( k \in (-n, \infty) \) to be a path starting at \( n + k \) and reaching \( (n + k) - 1 \) for the first time. Of course both the original and the shifted path have the same probability. Hence the probability of all the ways of going down 1 from \( n \) is the same as that from \( n + k \).

**STAFF NOTE:** Stimulate a brief discussion of why \( r \) is independent of \( n > 0 \).

(b) Conclude that if \( p \leq 1/2 \), then \( r = 1 \).

**Solution.** \( pr^2 - r + q \) has roots \( q/p \) and 1. So \( r = 1 \) or \( r = q/p \). But \( r \leq 1 \), which implies \( r = 1 \) when \( q/p \geq 1 \), that is, when \( p \leq 1/2 \).

In fact \( r = q/p \) when \( q/p < 1 \), namely, when \( p > 1/2 \), but this requires an additional argument that we omit.

(c) Prove that even in a fair game, the gambler is sure to get ruined *no matter how much money he starts with!*
Solution. The proof is by induction with hypothesis

\[ P(n) \equiv \Pr[\text{ruin starting with } \$n] = 1. \]

**base case** \((n = 0)\): If the stake is zero, the gambler is ruined at the start, so \( P(0) \) is true by definition.

**inductive step**: If the gambler’s initial stake is \( n \), the gambler will be ruined iff his stake gets reduced to \( n - 1 \) and he gets ruined after that. But by part (b), with probability 1 the gambler’s stake will be reduced to \( n - 1 \), and by induction hypothesis, he will then be ruined also with probability 1. Since the intersection of probability 1 events has probability 1, \( P(n) \) holds.

We conclude by induction that \( \forall n. P(n) \), as claimed.

\( \blacksquare \)

\((d)\) Let \( t \) be the expected time for the gambler’s stake to go down by 1 dollar. Verify that

\[ t = q + p(1 + 2t). \]

Conclude that starting with a 1 dollar stake in a fair game, the gambler can expect to play forever!

**Solution.** By Total Expectation

\[ t = \mathbb{E}[\text{#steps to be down } \$1 \mid \text{ lose the first bet}] \Pr[\text{lose the first bet}] + \mathbb{E}[\text{#steps to be down } \$1 \mid \text{ win the first bet}] \Pr[\text{win the first bet}] = q + p \mathbb{E}[\text{#steps to be down } \$1 \mid \text{ win the first bet}]. \]

But

\[ \mathbb{E}[\text{#steps to be down } \$1 \mid \text{ win the first bet}] = 1 + \mathbb{E}[\text{#steps to be down } \$2] = 1 + \mathbb{E}[\text{#steps to be down the first } \$1] + \mathbb{E}[\text{#steps to be down another } \$1] = 1 + 2t. \]

This implies the required formula \( t = q + p(1 + 2t) \). If \( p = 1/2 \) we conclude that \( t = 1 + t \), which means \( t \) must be infinite.

\( \blacksquare \)

**Problem 3.**
Let \( R \) be a positive integer valued random variable such that

\[ \text{PDF}_{R}(n) = \frac{1}{cn^3}, \]

where

\[ c \equiv \sum_{n=1}^{\infty} \frac{1}{n^3}. \]

(a) Prove that \( \mathbb{E}[R] \) is finite.
Solution.

\[
\text{Ex}[R] := \sum_{n \in \mathbb{N}^+} n \cdot \frac{1}{cn^3} = \sum_{n \in \mathbb{N}^+} \frac{1}{cn^2} < 1 + \int_1^\infty \frac{1}{cx^2} \, dx = 1 + \frac{1}{c} < \infty.
\]

\( \blacksquare \)

(b) Prove that \( \text{Var}[R] \) is infinite.

Solution. Since

\[
\text{Var}[R] = \text{Ex}[R^2] - \text{Ex}^2[R],
\]

and \( \text{Ex}^2[R] < \infty \) by part (a), we need only show that \( \text{Ex}[R^2] = \infty \). But

\[
\text{Ex}[R^2] := \sum_{n \in \mathbb{N}^+} n^2 \cdot \frac{1}{cn^3} = \sum_{n \in \mathbb{N}^+} \frac{1}{cn} = \frac{1}{c} \cdot \lim_{n \to \infty} H_n = \infty.
\]

\( \blacksquare \)

A joking way to phrase the point of this example is “The square root of infinity may be finite.” Namely, let \( T := R^2 \). Then the solution to part (b) implies that \( \text{Ex}[T] = \infty \) while \( \text{Ex}[^{\sqrt{T}}] < \infty \) by (a).

Problem 4.

You have a biased coin with nonzero probability \( p < 1 \) of tossing a Head. You toss until a Head comes up and record the number, \( k \), of Tails that preceded this first Head. Then you keep tossing until you get another run of tails of nearly the same length, namely, of length \( \max\{k - 10, 0\} \). Prove that the expected number of Heads you toss is infinite.

Solution. Let the random variable \( T \) be the length of your initial run of tails. If \( T = k \), then the expected number of Heads tossed until getting another run of Tails of length at least \( k_{10} := \min\{k - 10, 0\} \) will be the mean time to failure, where “failing” means tossing \( k_{10} \) Tails. Since the probability of failure is \( q^{k_{10}} \), where \( q := 1 - p \), this mean time is \( 1/q^{k_{10}} \). Letting \( H \) be the number of Heads tossed, we have

\[
\text{Ex}[H] = \sum_{k \in \mathbb{N}} \text{Ex}[H \mid T = k] \cdot \text{Pr}[T = k]
\]

\[
= \sum_{k \in \mathbb{N}} \frac{1}{q^{k_{10}}} \cdot q^k \cdot p
\]

\[
= \text{constant} + \sum_{k \geq 10} \frac{1}{q^{k-10}} \cdot q^k \cdot p
\]

\[
= \text{constant} + p \sum_{k \geq 10} q^{10} = \infty.
\]

\( \blacksquare \)