Staff Solutions to In-Class Problems Week 12, Mon.

STAFF NOTE: Linear Recurrences Generating Functions, Ch. 15.4

Carried over from Class Problems 11, Friday:

Problem 1.
We are interested in generating functions for the number of different ways to compose a bag of $n$ donuts subject to various restrictions. For each of the restrictions in parts (a)-(e) below, find a closed form for the corresponding generating function.

(a) All the donuts are chocolate and there are at least 3.

Solution. There are no ways to select 0, 1, or 2 donuts, and one way to select $n$ chocolate donuts for each $n > 2$, so the generating function is

$$x^3 + x^4 + x^5 + \cdots = x^3 \left(1 + x + x^2 + \cdots\right) = \frac{x^3}{1 - x}.$$

(b) All the donuts are glazed and there are at most 2.

Solution. There is one way to select 0, 1, or 2 glazed donuts, and no ways to select $n$ donuts for each $n > 2$, so the generating function is

$$1 + x + x^2.$$

(c) All the donuts are coconut and there are exactly 2 or there are none.

Solution.

$$1 + x^2$$

(d) All the donuts are plain and their number is a multiple of 4.

Solution. The generating function is

$$1 + x^4 + x^8 + \cdots + x^4n + \cdots = \sum_{i=0}^{\infty} (x^4)^n = \frac{1}{1 - x^4}.$$

(e) The donuts must be chocolate, glazed, coconut, or plain with the numbers of each flavor subject to the constraints above.
Solution. By the Convolution Rule, the generating function for selecting donuts with these constraints is the product of the preceding generating functions:

\[
\frac{x^3}{1-x} (1 + x + x^2)(1 + x^2) \frac{1}{1-x^4} = \frac{x^3(1 + x + x^2)(1 + x^2)}{(1-x)^2(1+x)(1+x^2)} = \frac{x^3(1 + x + x^2)}{(1-x)^2(1+x)}
\]

(f) Now find a closed form for the number of ways to select \(n\) donuts subject to the above constraints.

Solution. We would like to convert the generating function

\[
\frac{x^3(1 + x + x^2)}{(1-x)^2(1+x)}
\]

into partial fraction form. This requires that the numerator have lower degree than the denominator. We could accomplish this by expressing the ratio as a quotient and remainder, but in this case another simple approach applies. Namely, let

\[
G(x) := \frac{1 + x + x^2}{(1-x)^2(1+x)},
\]

so the generating function for donut selections is \(x^3G(x)\). Now we can express \(G(x)\) in partial fraction form and then use the fact that

\[
[x^n]x^3G(x) = [x^{n-3}]G(x)
\]

to obtain the generating function coefficients from the coefficients of \(G(x)\).

Expanding \(G(x)\) into partial fractions gives

\[
G(x) = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1+x}
\]  \hspace{1cm} (1)

for some constants, \(A, B, C\). We know that the coefficient of \(x^n\) in the series for \((1-x)^2\) is, by the Convolution Rule, the number of ways to select \(n\) items of two different kinds, namely, \(\binom{n+1}{1} = n+1\), so we conclude that the \(n\)th coefficient in the series for \(G(x)\) is

\[
A + B(n+1) + C(-1)^n.
\]  \hspace{1cm} (2)

To find \(A, B, C\), we multiply both sides of (1) by the denominator \((1-x)^2(1+x)\) to obtain

\[
1 + x + x^2 = A(1-x)(1+x) + B(1+x) + C(1-x)^2.
\]  \hspace{1cm} (3)

Letting \(x = 1\) in (3), we conclude that \(3 = 2B\), so \(B = 3/2\). Then, letting \(x = -1\), we conclude \((-1)^2 = C2^2\), so \(C = 1/4\). Finally, letting \(x = 0\), we have

\[
1 = A + B + C = A + \frac{3}{2} + \frac{1}{4},
\]

so \(A = -3/4\). Then from (2), we conclude that

\[
[x^n]G(x) = \frac{-3}{4} + \frac{3(n+1)}{2} + \frac{(-1)^n}{4} = \frac{6n + 3 + (-1)^n}{4}.
\]
So the \( n \)th coefficient in the series for the generating function, \( x^3G(x) \), for donut selections is zero for \( n < 3 \), and, for \( n \geq 3 \), is \([x^{n-3}]G(x)\), namely,
\[
\frac{6(n-3) + 3 + (-1)^{n-3}}{4} = \frac{6n - 15 - (-1)^n}{4}.
\]

**Problem 2.**

Less well-known than the Towers of Hanoi—but no less fascinating—are the Towers of Sheboygan. As in Hanoi, the puzzle in Sheboygan involves 3 posts and \( n \) disks of different sizes. Initially, all the disks are on post #1:

![Post Layout](image)

The objective is to transfer all \( n \) disks to post #2 via a sequence of moves. A move consists of removing the top disk from one post and dropping it onto another post with the restriction that a larger disk can never lie above a smaller disk. Furthermore, a local ordinance requires that a disk can be moved only from a post to the next post on its right—or from post #3 to post #1. Thus, for example, moving a disk directly from post #1 to post #3 is not permitted.

(a) One procedure that solves the Sheboygan puzzle is defined recursively: to move an initial stack of \( n \) disks to the next post, move the top stack of \( n - 1 \) disks to the furthest post by moving it to the next post two times, then move the big, \( n \)th disk to the next post, and finally move the top stack another two times to land on top of the big disk. Let \( s_n \) be the number of moves that this procedure uses. Write a simple linear recurrence for \( s_n \).

**Solution.**

\[
s_0 = 0, \\
\begin{align*}
s_n &= 2s_{n-1} + 1 + 2s_{n-1} = 4s_{n-1} + 1 \quad &\text{for } n > 0,
\end{align*}
\]

\( \blacksquare \)

(b) Let \( S(x) \) be the generating function for the sequence \( \langle s_0, s_1, s_2, \ldots \rangle \). Carefully show that
\[
S(x) = \frac{x}{(1-x)(1-4x)}.
\]

**Solution.**

\[
\begin{align*}
S(x) &= s_0 + s_1x + s_2x^2 + s_3x^3 + \cdots, \\
-4xS(x) &= -4s_0x - 4s_1x^2 - 4s_2x^3 - \cdots, \\
-1/(1-x) &= -1 - x - x^2 - x^3 - \cdots, \\
S(x)(1 - 4x) - \frac{1}{1-x} &= -1 + 0x + 0x^2 + 0x^3 + \cdots, \\
&= -1.
\end{align*}
\]
so 
\[ S(x)(1 - 4x) - \frac{1}{1 - x} = -1, \]
and 
\[ S(x) = \frac{x}{(1 - x)(1 - 4x)}. \]

(c) Give a simple formula for \( s_n \).

**Solution.** We can express \( x/(1 - x)(1 - 4x) \) using partial fractions as
\[ \frac{x}{(1 - x)(1 - 4x)} = \frac{a}{1 - x} + \frac{b}{1 - 4x} \] (5)
for some constants \( a, b \). Multiplying both sides of (5) by the left hand denominator yields
\[ x = a(1 - 4x) + b(1 - x). \] (6)
Letting \( x = 1 \) yields \( a = -1/3 \) and letting \( x = 1/4 \) yields \( b = 1/3 \). Now from (5), we have
\[ S(x) = \frac{-1/3}{1 - x} + \frac{1/3}{1 - 4x} \]
so
\[ s_n = -\frac{1}{3} + \frac{1}{3} \cdot 4^n = \frac{4^n - 1}{3}. \]

**Problem 3.**
The famous mathematician, Fibonacci, has decided to start a rabbit farm to fill up his time while he’s not making new sequences to torment future college students. Fibonacci starts his farm on month zero (being a mathematician), and at the start of month one he receives his first pair of rabbits. Each pair of rabbits takes a month to mature, and after that breeds to produce one new pair of rabbits each month. Fibonacci decides that in order never to run out of rabbits or money, every time a batch of new rabbits is born, he’ll sell a number of newborn pairs equal to the total number of pairs he had three months earlier. Fibonacci is convinced that this way he’ll never run out of stock.

(a) Define the number, \( r_n \), of pairs of rabbits Fibonacci has in month \( n \), using a recurrence relation. That is, define \( r_n \) in terms of various \( r_i \) where \( i < n \).

**Solution.** According to the description above, \( r_0 = 0 \) and \( r_1 = 1 \). Since the rabbit pair received at the first month is too young to breed, \( r_2 = 1 \) as well. After that, \( r_n \) is equal to the number, \( r_{n-1} \), of rabbit pairs in the previous month, plus the number of newborn pairs, minus the number, \( r_{n-3} \), he sells. The number of newborn pairs equals to the number of breeding pairs from the previous month, which is precisely the total number, \( r_{n-2} \), of pairs from two months before.
Thus,
\[ r_n = r_{n-1} + (r_{n-2} - r_{n-3}). \]
(b) Let \( R(x) \) be the generating function for rabbit pairs,

\[
R(x) := r_0 + r_1 x + r_2 x^2 + \cdots
\]

Express \( R(x) \) as a quotient of polynomials.

**Solution.** Reasoning as in the derivation of the generating function for the original Fibonacci numbers, we have

\[
\begin{align*}
R(x) &= r_0 + r_1 x + r_2 x^2 + r_3 x^3 + r_4 x^4 + \cdots, \\
-xR(x) &= -r_0 x - r_1 x^2 - r_2 x^3 - r_3 x^4 - \cdots, \\
-x^2 R(x) &= -r_0 x^2 - r_1 x^3 - r_2 x^4 - \cdots, \\
x^3 R(x) &= + r_0 x^3 + r_1 x^4 - \cdots.
\end{align*}
\]

so

\[
R(x)(1 - x - x^2 + x^3) = r_0 + (r_1 - r_0) x + (r_2 - r_1 - r_0) x^2 + 0 x^3 + 0 x^4 + \cdots,
\]

and

\[
R(x) = \frac{x}{1 - x - x^2 + x^3} = \frac{x}{(1 + x)(1 - x)^2}.
\]

(c) Find a partial fraction decomposition of the generating function \( R(x) \).

**Solution.** We know

\[
R(x) = \frac{A}{1 + x} + \frac{B}{1 - x} + \frac{C}{(1 - x)^2}
\]

for some numbers \( A, B, C \). Multiplying both sides of this equation by \((1 + x)(1 - x)^2\) gives

\[
x = A(1 - x)^2 + B(1 + x)(1 - x) + C(1 + x).
\]

Letting \( x = 1 \) gives \( C = 1/2 \), letting \( x = -1 \) gives \( A = -1/4 \), and letting \( x = 0 \) then gives \( B = -(A + C) = -1/4 \).

(d) Finally, use the partial fraction decomposition to come up with a closed form expression for the number of pairs of rabbits Fibonacci has on his farm on month \( n \).

**Solution.** We find the coefficient as the sum of the coefficients for each term in the partial fraction expansion.

\[
\begin{align*}
A/(1 + x) &= -1/4 - (1/4)(-x) - \cdots - (1/4)(-x)^n - \cdots,

B/(1 - x) &= -1/4 - (1/4)x - \cdots - (1/4)x^n - \cdots,

C/(1 - x)^2 &= 1/2 + (2/2)x + \cdots + ((n + 1)/2)x^n + \cdots,
\end{align*}
\]

so

\[
R(x) = 1x + 1x^2 + \cdots + \left(\frac{n + 1}{2} - \frac{(-1)^n + 1}{4}\right)x^n + \cdots,
\]

and

\[
r_n = \left\lfloor \frac{n}{2} \right\rfloor.
\]

If you didn’t finish Problem 1 above last Friday, you may not have time to finish the following problem in class. Be sure to study the solution in any case.
Problem 4.
Alyssa Hacker sends out a video that spreads like wildfire over the UToob network. On the day of the release—call it day zero—and the day following—call it day one—the video doesn’t receive any hits. However, starting with day two, the number of hits, \( r_n \), can be expressed as seven times the number of hits on the previous day, four times the number of hits the day before that, and the number of days that has passed since the release of the video plus one. So, for example on day 2, there will be \( 7 \times 0 + 4 \times 0 + 3 = 3 \) hits.

(a) Give a linear a recurrence for \( r_n \).

Solution.

\[
\begin{align*}
  r_n &= \begin{cases} 
    7r_{n-1} + 4r_{n-2} + (n + 1) & \text{for } n \geq 2, \\
    0 & \text{for } n < 2.
  \end{cases}
\]

(b) Express the generating function \( R(x) := \sum_{n=0}^{\infty} r_n x^n \) as a quotient of polynomials or products of polynomials. You do not have to find a closed form for \( r_n \).

Solution. Using the formula \((15.13)\) for the coefficient of \((1 - x)^{-2}\), we have

\[
\begin{align*}
  R(x) &= r_0 + r_1 x + r_2 x^2 + r_3 x^3 + \cdots \\
  -7x R(x) &= -7r_0 x - 7r_1 x^2 - 7r_2 x^3 - \cdots \\
  -4x^2 R(x) &= -4r_0 x^2 - 4r_1 x^3 - \cdots \\
  -(1 - x)^{-2} &= 1 - 1 - 2x + 3x^2 - 4x^3 + \cdots \\
  &= r_0 - 1 - (r_1 - 7r_0 - 2)x + 0x^2 + 0x^3 + \cdots \\
  &= -1 + -2x + 0x^2 + 0x^3 + \cdots.
\end{align*}
\]

Therefore,

\[
R(x)(1 - 7x - 4x^2) - \frac{1}{(1 - x)^2} = -(1 + 2x),
\]

so

\[
R(x) = \frac{x^2(3 + 2x)}{(1 - 7x - 4x^2)(1 - x)^2}.
\]