Staff Solutions to Problem Set 8

Reading: Chapter 13. Sums and Products, Asymptotics, omitting 13.6 & 13.7.5; Chapter 14. through Section 14.5 on Counting with Bijection, Generalized Product and Division Rules

Problem 1.
Prove that $\sum_{k=1}^{n} k^6 = \Theta(n^7)$.

Solution. Let $S_n := \sum_{k=1}^{n} k^6$.

One approach is to use the Integral Method:

$$\frac{n^7}{7} = \int_{0}^{n} x^6 \, dx \leq S_n \leq \int_{0}^{n} (x + 1)^6 \, dx = \frac{(n + 1)^7}{7} - \frac{1}{7}.$$ 

So we have $n^7 \leq 7S_n$, and so $n^7 = O(S_n)$. Also $(n + 1)^7/7 - 1/7 = O(n^7)$, and so $S_n = O(n^7)$. Hence, $S_n = \Theta(n^7)$.

An alternative approach not using the Integral Method goes as follows. There are $n$ terms in $S_n$ and each term is at most $n^6$, so $S_n \leq n \cdot n^6 = n^7 = O(n^7)$. So $S_n = O(n^7)$.

On the other hand, at least $(n-1)/2$ of the terms are as large as $[(n-1)/2]^6$, so

$$S_n \geq \left(\frac{n-1}{2}\right) \cdot \left\lfloor\frac{(n-1)/2}{6}\right\rfloor$$

$$= \left[\frac{(n-1)/2}{6}\right]^7$$

$$\geq (n/3)^7$$

for $n > 3$, so $n^7 \leq 3^7 \cdot S_n$. In other words, $n^7 = O(S_n)$.

Problem 2.
Indicate which of the following holds for each pair of functions $(f(n), g(n))$ in the table below. Assume $k \geq 1$, $c > 0$, and $c > 0$ are constants. Pick the four table entries you consider to be the most challenging or interesting and justify your answers to these.

<table>
<thead>
<tr>
<th>$f(n)$</th>
<th>$g(n)$</th>
<th>$f = O(g)$</th>
<th>$f = o(g)$</th>
<th>$g = O(f)$</th>
<th>$g = o(f)$</th>
<th>$f = \Theta(g)$</th>
<th>$f \sim g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2^n$</td>
<td>$2^{n/2}$</td>
<td>no</td>
<td>no</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>$\sqrt{n}$</td>
<td>$n^{\sin(n\pi/2)}$</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>$\log(n!)$</td>
<td>$\log(n^n)$</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
<td>no</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>$n^k$</td>
<td>$c^n$</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>$\log^n n$</td>
<td>$n^c$</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>no</td>
</tr>
</tbody>
</table>

Solution.

<table>
<thead>
<tr>
<th>$f(n)$</th>
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</tr>
</thead>
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<td>$n^c$</td>
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<td>yes</td>
<td>no</td>
<td>no</td>
<td>no</td>
<td>no</td>
</tr>
</tbody>
</table>

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Following are some hints on deriving the table above:

(a) \( \frac{2^n}{2^{n/2}} = 2^{n/2} \) grows without bound as \( n \) grows—it is not bounded by a constant.

(b) When \( n \) is even, then \( n^{\sin(n\pi/2)} = 1 \). So, no constant times \( n^{\sin(n\pi/2)} \) will be an upper bound on \( \sqrt{n} \) as \( n \) ranges over even numbers. When \( n \equiv 1 \mod 4 \), then \( n^{\sin(n\pi/2)} = n^1 = n \). So, no constant times \( \sqrt{n} \) will be an upper bound on \( n^{\sin(n\pi/2)} \) as \( n \) ranges over numbers \( \equiv 1 \mod 4 \).

(c) 
\[
\log(n!) = \log\left(2\pi n \left(\frac{n}{e}\right)^n \pm c_n\right) \\
= \log n + n(\log n - 1) \pm d_n \\
\sim n \log n \\
= \log n^n.
\]

where \( a \leq c_n, d_n \leq b \) for some constants \( a, b \in \mathbb{R} \) and all \( n \). Here equation (1) follows by taking logs of Stirling’s formula, (2) follows from the fact that the log of a product is the sum of the logs, and (3) follows because any constant, \( \log n \), and \( n \) are all \( o(n \log n) \) and hence so is their sum.

(d) *Polynomial* growth versus *exponential* growth.

(e) *Polylogarithmic* growth versus *polynomial* growth.

Problem 3.
In a standard 52-card deck, each card has one of thirteen ranks in the set, \( R \), and one of four suits in the set, \( S \), where

\[
R := \{A, 2, \ldots, 10, J, Q, K\}, \\
S := \{\spadesuit, \heartsuit, \clubsuit, \diamondsuit\}.
\]

A 5-card *hand* is a set of five distinct cards from the deck.

For each part describe a bijection between a set that can easily be counted using the Product and Sum Rules of Ch. 14.1, and the set of hands matching the specification. *Give bijections, not numerical answers.*

For instance, consider the set of 5-card hands containing all 4 suits. Each such hand must have 2 cards of one suit. We can describe a bijection between such hands and the set \( S \times R_2 \times R_3 \) where \( R_2 \) is the set of two-element subsets of \( R \). Namely, an element

\[
(s, \{r_1, r_2\}, (r_3, r_4, r_5)) \in S \times R_2 \times R_3
\]

indicates

1. the repeated suit, \( s \in S \),
2. the set, \( \{r_1, r_2\} \in R_2 \), of ranks of the cards of suit, \( s \), and
3. the ranks \( (r_3, r_4, r_5) \) of the remaining three cards, listed in increasing suit order where \( \heartsuit < \diamond < \clubsuit < \spadesuit \).

For example,

\[
(\spadesuit, \{10, A\}, (J, J, 2)) \leftrightarrow \{A\spadesuit, 10\spadesuit, J\diamond, J\clubsuit, 2\spadesuit\}.
\]
(a) A single pair of the same rank (no 3-of-a-kind, 4-of-a-kind, or second pair).

Solution. There is a bijection with the set $R_4 \times \{1, 2, 3, 4\} \times S_2 \times S^3$ where an element

$$(r_1, r_2, r_3, r_4), i, (s_1, s_2), (s_1, s_2, s_3) \in R_4 \times \{1, 2, 3, 4\} \times S_2 \times S^3$$

specifies

1. the 4 ranks among the five cards in the hand,
2. the position, $i$, of the rank of the pair when the four ranks are listed in increasing order,
3. the set of two suits of the pair,
4. the sequence, $(s_1, s_2, s_3)$, of suits of the unpaired cards, in rank order.

For example,

$$(\{3, J, Q, 2\}, 4, \{\heartsuit, \spadesuit\}, (\heartsuit, \diamondsuit, \spadesuit)) \leftrightarrow \{Q \heartsuit, Q \spadesuit, 2 \heartsuit, 3 \diamondsuit, J \spadesuit\}.$$  

(b) Three or more aces.

Solution. There is a bijection between the hands with exactly 3 aces along with two non-Ace cards of different ranks, $r_1, r_2$ and the set, $A_3 := S \times (R - \{A\})_2 \times S \times S$, of sequences of the form $(s, \{r_1, r_2\}, s_1, s_2)$ specifying

1. the suit, $s$, of the missing Ace,
2. the ranks of the non-Ace cards,
3. suit of the lower-rank non-Ace,
4. suit of the higher-rank non-Ace.

Similarly, there is a bijection between the hands with exactly 3 aces along with a pair of non-Ace cards of the same rank, and the set, $A_{\text{full}} := S \times (R - \{A\}) \times S_2$, of sequences of the form $(s, r, \{s_1, s_2\})$ specifying

1. the suit, $s$, of the missing Ace,
2. the rank, $r$, of the pair of non-Ace cards,
3. set $\{s_1, s_2\}$ of two suits of the non-Ace pair.

Finally, there is a bijection between the hands with exactly 4 aces and the set, $A_4 := (R - \{A\}) \times S$, of sequences of the form $(r, s)$ specifying the rank, $r$, and the suit, $s$, of the non-Ace.

$A_3$, $A_{\text{full}}$, and $A_4$ are disjoint. Since the set of hands with exactly 3 aces two cards of different rank, the set of “Aces-full” hands (3 Aces and a pair or the same rank), and the set of hands with exactly 4 aces are disjoint as well, we have described a bijection between the hands with three or more aces and the set $A_3 \cup A_{\text{full}} \cup A_4$.  

Problem 4.

Answer the following questions with a number or a simple formula involving factorials and binomial coefficients. Briefly explain your answers.

(a) How many ways are there to order the 26 letters of the alphabet so that no two of the vowels $a, e, i, o, u$ appear consecutively and the last letter in the ordering is not a vowel?

Hint: Every vowel appears to the left of a consonant.
Solution. The constraint on where vowels can appear is equivalent to the requirement that every vowel appears to the left of a consonant. So given a sequence of the 21 consonants, there are \( \binom{21}{5} \) positions where the 5 vowels can be placed. After determining such a placement, we can reorder the consonants and vowels in any order. Thus, the number is:

\[
\binom{21}{5} \cdot 21! \cdot 5!.
\]

(b) How many ways are there to order the 26 letters of the alphabet so that there are at least two consonants immediately following each vowel?

Solution. The pattern of consonants and vowels in any permutation of the 26 letters of the alphabet can be indicated by a binary string with 5 ones indicating where the vowels occur and 21 zeros where the consonants occur. Patterns where every vowel has at least two consonants to its right can be constructed by taking a sequence of 16 zeros and inserting “10” to the left of 5 of the 16 zeros. There are \( \binom{16}{5} \) ways to do this. For any such pattern, there are 5! ways to place the vowels in the positions where ones occur and 21! ways to place the consonants where the zeroes occur. Thus, the final answer is:

\[
\binom{16}{5} \cdot 5! \cdot 21!.
\]

(c) In how many different ways can \( 2n \) students be paired up?

Solution. Pair up students by the following procedure. Line up the students and pair the first and second, the third and fourth, the fifth and sixth, etc. The students can be lined up in \( (2n)! \) ways. However, this overcounts by a factor of \( 2^n \), because we would get the same pairing if the first and second students were swapped, the third and fourth were swapped, etc. Furthermore, we are still overcounting by a factor of \( n! \), because we would get the same pairing even if pairs of students were permuted, e.g. the first and second were swapped with the ninth and tenth. Therefore, the number of pairings is:

\[
\frac{(2n)!}{2^n \cdot n!}
\]

(d) Two \( n \)-digit sequences of digits 0,1,...,9 are said to be of the same type if the digits of one are a permutation of the digits of the other. For \( n = 8 \), for example, the sequences 03088929 and 00238899 are the same type. How many types of \( n \)-digit integers are there?

Solution. The type of a string is determined by the numbers of occurrences of the 9 different digits in the string. So there is a bijection between types of strings and strings with \( n \) 0’s and nine 1’s: the length of the block of 0’s before the \( i \)th 1 equals the number of occurrences of the digit \( i \) (and the length of the final block of 0’s equals the number of occurrences of the digit 9). Therefore, the number of different types is \( \binom{n+9}{9} \).