Staff Solutions to Mini-Quiz 6

Problem 1 (6 points).

(a) Circle all the properties below that are preserved under graph isomorphism.

- There is a cycle that includes all the vertices.
- Two edges are of equal length.
- The graph remains connected if any two edges are removed.
- There exists an edge that is an edge of every spanning tree.
- The negation of a property that is preserved under isomorphism.

Solution. All but the second one are preserved. ■

(b) For the following statements about finite trees, circle true or false, and provide counterexamples for those that are false.

- Any connected subgraph is a tree. true false
  Solution. true. ■

- Adding an edge between two nonadjacent vertices creates a cycle. true false
  Solution. true. ■

- The number of vertices is one less than twice the number of leaves. true false
  Solution. false. This property holds for full binary trees, but not in general. A tree with two vertices is a counterexample. ■

- The number of vertices is one less than the number of edges. true false
  Solution. false. This got “edges” and “vertices” reversed. Every tree is a counterexample. ■

- For every finite graph (not necessarily a tree), there is one (a finite tree) that spans it. true false
  Solution. false. Any disconnected graph is a counterexample. ■

Problem 2 (4 points).
Show that

\[ \sum_{i=1}^{\infty} i^p \]

converges to a finite value iff \( p < -1 \).
Solution. The sum is $\Theta \left( \int_1^\infty x^p \, dx \right)$. For $p \neq -1$, the indefinite integral is $\frac{x^{p+1}}{p+1}$.

- If $p < -1$, then $p + 1 < 0$, so $\lim_{x \to \infty} x^{p+1} = 0$, and the definite integral from 1 to $\infty$ evaluates to $-1/(p + 1)$. Hence the sum is bounded from above, and since it is increasing, it has a finite limit, that is, it converges.

- If $p > -1$, then $p + 1 > 0$, so $\lim_{x \to \infty} x^{p+1} = \infty$, and the definite integral diverges.

- For $p = -1$ the indefinite integral is $\log x$ which also approaches $\infty$ as $x$ approaches $\infty$, so the definite integral also diverges.

Also: for $p = -1$, the sum is the harmonic series which we know does not converge. Since the term $i^p$ is increasing in $p$ for $i > 1$, the sum will be larger, and hence also diverge for $p > -1$. 

$\blacksquare$