Staff Solutions to Mini-Quiz 3

Problem 1 (3 points).
Let \( A \) be the set containing the five sets: \( \{a\}, \{b, c\}, \{b, d\}, \{a, e\}, \{e, f\} \), and let \( B \) be the set containing the three sets: \( \{a, b\}, \{b, c, d\}, \{e, f\} \). Let \( R \) be the “is subset of” binary relation from \( A \) to \( B \) defined by the rule:
\[
X R Y \iff X \subseteq Y.
\]

(a) Fill in the arrows so the following figure describes the graph of the relation, \( R \):

\[
\begin{array}{c|c}
A & \text{arrows} & B \\
\{a\} & \{a, b\} & \\
\{b, c\} & \{b, c, d\} & \\
\{b, d\} & \{e, f\} & \\
\{a, e\} & & \\
\{e, f\} & & \\
\end{array}
\]

Solution. Four arrows for \( R \):
\[
\begin{align*}
\{a\} & \subset \{a, b\} \\
\{b, c\} & \subset \{b, c, d\} \\
\{b, d\} & \subset \{b, c, d\} \\
\{e, f\} & \subset \{e, f\}
\end{align*}
\]

(b) Circle the properties below possessed by the relation \( R \):

<table>
<thead>
<tr>
<th>FUNCTION</th>
<th>TOTAL</th>
<th>INJECTIVE</th>
<th>SURJECTIVE</th>
<th>BIJECTIVE</th>
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Solution. From part (b), the “is subset of” relation, \( R \), is a surjective function.

(c) Circle the properties below possessed by the relation \( R^{-1} \):

<table>
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<th>TOTAL</th>
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<th>SURJECTIVE</th>
<th>BIJECTIVE</th>
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</table>

Solution. From part (c), the inverse relation, \( R^{-1} \), is a total injection.

Problem 2 (7 points).
Any amount of ten or more cents postage that is a multiple of five can be made using only 10¢ and 15¢ stamps. Prove this by induction (ordinary or strong, but say which) using the induction hypothesis

\[ S(n) \ ::= \ (5n + 10)¢ \text{ postage can be made using only 10¢ and 15¢ stamps.} \]

Solution. Proof. The proof is by strong induction.

Base case \( n = 0 \): \( 5 \cdot 0 + 10 = 10¢ \text{ postage can be made with one 10¢ stamp.} \)

Inductive step: We assume \( n \geq 0 \) and the hypothesis \( S(n) \) to prove \( S(n + 1) \). The proof is by cases:

- case \( n = 0 \): In this case \( 5(n + 1) + 10 = 15 \). So \( S(n + 1) \) holds because 15¢ postage can be made using one 15¢ stamp.

- case \( n > 0 \): Since \( n - 1 \geq 0 \), we know by strong induction that \( S(n - 1) \) holds. So we can make \( 5(n - 1) + 10¢ \) postage. Adding a 10¢ stamp yields \( 5(n - 1) + 10 + 10 = 5(n + 1) + 10¢ \) postage, which proves \( S(n + 1) \).

Since \( S(n + 1) \) holds in any case, the inductive step has been proved.

It follows by induction that \( S(n) \) holds for all \( n \in \mathbb{N} \), that is, starting at 10¢, every multiple of 5¢ postage can be made with 10¢ and 15¢ stamps.