Final Examination

Your name:__________________________________________

- This exam is closed book except for a 4-sided crib sheet. Total time is 3 hours.
- Write your solutions in the space provided with your name on every page. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem’s page.
- GOOD LUCK!

<table>
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<th>Points</th>
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DO NOT WRITE BELOW THIS LINE

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Problem 1 (Numbers Short Answer) (8 points).
Give counterexamples for each of the statements below that are false.

(a) For integers $a$ and $b$ there are integers $x$ and $y$ such that: $ax + by = 1$

(b) $\gcd(mb + r, b) = \gcd(r, b)$ for all integers $m, r$ and $b$.

(c) For every prime $p$ and every integer $k$, $k^{p-1} \equiv 1 \pmod p$.

(d) For primes $p \neq q$, $\phi(pq) = (p - 1)(q - 1)$, where $\phi$ is Euler’s totient function.

(e) Suppose $a, b, c, d \in \mathbb{N}$ and $a$ and $b$ are relatively prime to $d$. Then

$$[ac \equiv bc \pmod{d}] \ \text{IMPLIES} \ \ [a \equiv b \pmod{d}].$$
 Problem 2 (Relations Short Answer) (7 points).
Indicate which of the following relations below are equivalence relations, (E), strict partial orders (S), weak partial orders (W), or none of the above (N).

For the partial orders, also indicate whether it is path-total (T).

(a) The superset relation, $\supseteq$ on the power set of the integers. 

(b) The relation $\text{Ex}[R] < \text{Ex}[S]$ between real-valued random variables $R, S$. 

(c) The divides relation on the positive powers of 4. 

For the next parts, let $f, g$ be nonnegative functions from the integers to the real numbers.

(d) $f = o(g)$, the “little Oh” relation.

(e) $f = O(g)$, the “big Oh” relation.

(f) $f = \Theta(g)$, the “Theta” relation.

(g) $f = O(g)$ AND NOT ($g = O(f)$).
Problem 3 (Law of Large Numbers Quantifiers) (6 points).
Let $G_1, G_2, G_3, \ldots$, be an infinite sequence of pairwise independent random variables with the same expectation, $\mu$, and the same finite variance. Let

$$f(n, \epsilon) := \Pr \left[ \left| \frac{\sum_{i=1}^{n} G_i}{n} - \mu \right| \leq \epsilon \right].$$

The Weak Law of Large Numbers can be expressed as a logical formula of the form:

$$\forall \epsilon > 0 \, Q_1 \, Q_2 \ldots \, [f(n, \epsilon) \geq 1 - \delta]$$

where $Q_1 \, Q_2 \ldots$ is a sequence of quantifiers from among:

$$\forall n \quad \exists n \quad \forall n_0 \quad \exists n_0 \quad \forall n \geq n_0 \quad \exists n \geq n_0 \quad \forall \delta > 0 \quad \exists \delta > 0 \quad \forall \delta \geq 0 \quad \exists \delta \geq 0$$

Here the $n$ and $n_0$ range over nonnegative integers, and $\delta$ and $\epsilon$ range over real numbers.

Write out the proper sequence $Q_1 \, Q_2 \ldots$.
Problem 4 (Binary Relations on 0,1) (8 points).
How many binary relations are there on the set {0, 1}?

How many are there that are:
  weak partial orders?

  equivalence relations?

  transitive?
Problem 5 (Modular Exponentiation) (7 points).

(a) What is the probability that an integer from 1 to 360 selected with uniform probability is relatively prime to 360?

(b) What is the value of $\text{rem}(7^{98}, 360)$?
Problem 6 (Bogus Coloring) (8 points).

**False Claim.** Let $G$ be a graph whose vertex degrees are all $\leq k$. If $G$ has a vertex of degree strictly less than $k$, then $G$ is $k$-colorable.

(a) Give a counterexample to the False Claim when $k = 2$.

(b) Underline the exact sentence or part of a sentence that is the first unjustified step in the following bogus proof of the False Claim.

_Bogus proof._ Proof by induction on the number $n$ of vertices:

The induction hypothesis, $P(n)$ is:

Let $G$ be an $n$-vertex graph whose vertex degrees are all $\leq k$. If $G$ also has a vertex of degree strictly less than $k$, then $G$ is $k$-colorable.

**Base case:** $(n = 1)$ $G$ has one vertex, the degree of which is 0. Since $G$ is 1-colorable, $P(1)$ holds.

**Inductive step:** We may assume $P(n)$. To prove $P(n + 1)$, let $G_{n+1}$ be a graph with $n + 1$ vertices whose vertex degrees are all $k$ or less. Also, suppose $G_{n+1}$ has a vertex, $v$, of degree strictly less than $k$. Now we only need to prove that $G_{n+1}$ is $k$-colorable.

To do this, first remove the vertex $v$ to produce a graph, $G_n$, with $n$ vertices. Let $u$ be a vertex that is adjacent to $v$ in $G_{n+1}$. Removing $v$ reduces the degree of $u$ by 1. So in $G_n$, vertex $u$ has degree strictly less than $k$. Since no edges were added, the vertex degrees of $G_n$ remain $\leq k$. So $G_n$ satisfies the conditions of the induction hypothesis, $P(n)$, and so we conclude that $G_n$ is $k$-colorable.

Now a $k$-coloring of $G_n$ gives a coloring of all the vertices of $G_{n+1}$, except for $v$. Since $v$ has degree less than $k$, there will be fewer than $k$ colors assigned to the nodes adjacent to $v$. So among the $k$ possible colors, there will be a color not used to color these adjacent nodes, and this color can be assigned to $v$ to form a $k$-coloring of $G_{n+1}$.

■
(e) With a slightly strengthened condition, the preceding proof of the False Claim could be revised into a sound proof of the following Claim:

**Claim.** Let $G$ be a graph whose vertex degrees are all $\leq k$. If (statement inserted from below) has a vertex of degree strictly less than $k$, then $G$ is $k$-colorable.

Circle each of the statements below that could be inserted to make the proof correct.

- $G$ is connected and
- $G$ has no vertex of degree zero and
- $G$ does not contain a complete graph on $k$ vertices and
- every connected component of $G$
- some connected component of $G$
**Problem 7 (Stable Matching) (7 points).**

Four unfortunate children want to be adopted by four foster families of ill repute. A child can only be adopted by one family, and a family can only adopt one child. Here are their preference rankings (most-favored to least-favored):

<table>
<thead>
<tr>
<th>Child</th>
<th>Families</th>
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<tbody>
<tr>
<td>Bottlecap:</td>
<td>Hatfields, McCoys, Grinches, Scrooges</td>
</tr>
<tr>
<td>Lucy:</td>
<td>Grinches, Scrooges, McCoys, Hatfields</td>
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<tr>
<td>Dingdong:</td>
<td>Hatfields, Scrooges, Grinches, McCoys</td>
</tr>
<tr>
<td>Zippy:</td>
<td>McCoys, Grinches, Scrooges, Hatfields</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Family</th>
<th>Children</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grinches:</td>
<td>Zippy, Dingdong, Bottlecap, Lucy</td>
</tr>
<tr>
<td>Hatfields:</td>
<td>Zippy, Bottlecap, Dingdong, Lucy</td>
</tr>
<tr>
<td>Scrooges:</td>
<td>Bottlecap, Lucy, Dingdong, Zippy</td>
</tr>
<tr>
<td>McCoys:</td>
<td>Lucy, Zippy, Bottlecap, Dingdong</td>
</tr>
</tbody>
</table>

(a) Exhibit two different stable matching of Children and Families.

<table>
<thead>
<tr>
<th>Family</th>
<th>Child in 1st match</th>
<th>Child in 2nd match</th>
</tr>
</thead>
<tbody>
<tr>
<td>Grinches:</td>
<td></td>
<td></td>
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<tr>
<td>Hatfields:</td>
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<td>Scrooges:</td>
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<tr>
<td>McCoys:</td>
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(b) Explain why the matchings of part (a) are the only two possible stable matchings between Children and Families.
Problem 8 (Generating Boat Trips) (8 points).
T-Pain is planning an epic boat trip and he needs to decide what to bring with him.

- He must bring some burgers, but they only come in packs of 6.
- He and his two friends can’t decide whether they want to dress formally or casually. He’ll either bring 0 pairs of flip flops or 3 pairs.
- He doesn’t have very much room in his suitcase for towels, so he can bring at most 2.
- In order for the boat trip to be truly epic, he has to bring at least 1 nautical-themed pashmina afghan.

(a) Let \( B(x) \) be the generating function for the number of ways to bring \( n \) burgers, \( F(x) \) for the number of ways to bring \( n \) pairs of flip flops, \( T(x) \) for towels, and \( A(x) \) for Afghans. Write simple formulas for each of these.

\[
B(x) = \quad \quad F(x) =
\]
\[
T(x) = \quad \quad A(x) =
\]

(b) Let \( g_n \) be the the number of different ways for T-Pain to bring \( n \) items (burgers, pairs of flip flops, towels, and/or afghans) on his boat trip. Let \( G(x) \) be the generating function \( \sum_{n=0}^{\infty} g_n x^n \). Verify that

\[
G(x) = \frac{x^7}{(1 - x)^2}.
\]

(c) Find a simple formula for \( g_n \).
Problem 9 (Random Sampling) (8 points).
You work for the president and you want to estimate the fraction $p$ of voters in the entire nation that will prefer him in the upcoming elections. You do this by random sampling. Specifically, you select a random voter and ask them who they are going to vote for. You do this $n$ times, with each voter selected with uniform probability and independently of other selections. Finally, you use the fraction $P$ of voters who said they will vote for the President as an estimate for $p$.

(a) Our theorems about sampling and distributions allow us to calculate how confident we can be that the random variable, $P$, takes a value near the constant, $p$. This calculation uses some facts about voters and the way they are chosen. Circle the true facts among the following:

1. Given a particular voter, the probability of that voter preferring the President is $p$.
2. The probability that some voter is chosen more than once in the random sample goes to one as $n$ increases.
3. The probability that some voter is chosen more than once in the random sample goes to zero as the population of voters grows.
4. All voters are equally likely to be selected as the third in the random sample of $n$ voters (assuming $n \geq 3$).
5. The probability that the second voter in the random sample will favor the President, given that the first voter prefers the President, is greater than $p$.
6. The probability that the second voter in the random sample will favor the President, given that the second voter is from the same state as the first, may not equal $p$.

(b) Suppose that according to your calculations, the following is true about your polling:

$$\text{Pr}[|P - p| \leq 0.04] \geq 0.95.$$ 

You do the asking, you count how many said they will vote for the President, you divide by $n$, and find the fraction is 0.53. Among the following, circle the legitimate things you might say in a call to the President:

1. Mr. President, $p = 0.53$!
2. Mr. President, with probability at least 95 percent, $p$ is within 0.04 of 0.53.
3. Mr. President, either $p$ is within 0.04 of 0.53 or something very strange (5-in-100) has happened.
4. Mr. President, we can be 95% confident that $p$ is within 0.04 of 0.53.
Problem 10 (Poker Deviation) (11 points).
A gambler plays 120 hands of draw poker, 60 hands of black jack, and 20 hands of stud poker per day. He wins a hand of draw poker with probability 1/6, a hand of black jack with probability 1/2, and a hand of stud poker with probability 1/5.

(a) What is the expected number of hands the gambler wins in a day?

(b) What would the Markov bound be on the probability that the gambler will win at least 108 hands on a given day?

(c) Assume the outcomes of the card games are pairwise, but possibly not mutually, independent. What is the variance in the number of hands won per day? You may answer with a numerical expression that is not completely evaluated.

(d) What would the Chebyshev bound be on the probability that the gambler will win at least 108 hands on a given day? You may answer with a numerical expression that is not completely evaluated.

(e) Assuming outcomes of the card games are mutually independent, show that the probability that the gambler will win at least 108 hands on a given day is much smaller than the bound in part (d). Hint: $e^{1 - 2 \ln 2} \leq 0.7$
Problem 11 (Structural Induction Congruence) (6 points).
The set $A\text{exp}$ of Arithmetic Expressions in the variable $x$ was defined recursively: expressions consisting solely of the variable $x$ or an arabic numeral, $k$, were the base cases, and the constructors were forming the sum, $[ e_1 + e_2 ]$, product, $[ e_1 \times e_2 ]$, or minus $- [ e_1 ]$ of $A\text{exp}$’s $e_1, e_2$. Then the value $\text{eval}(e, n)$ of an $A\text{exp}$ $e$ when the variable $x$ is equal to the integer $n$ has an immediate recursive definition based on the definition of $A\text{exp}$’s.

Prove by structural induction that for all $A\text{exp}$’s $e$,

$$\forall m, n, d \in \mathbb{Z}, d > 1. [ m \equiv n \pmod{d} ] \text{ IMPLIES } [\text{eval}(e, m) \equiv \text{eval}(e, n) \pmod{d}].$$  \hfill (1)

*Hint:* Be sure to consider both base cases. The proofs for the three constructors are very similar, so just write out the case for the sum constructor.
Problem 12 (Counting Paths) (8 points).

(a) Theory Hippo wants to travel from point (0, 0) to point (32, 43) using only steps that add one to the $x$ or the $y$ coordinate. Write a simple formula for the number of paths he can follow. (You may leave factorials and binomial coefficients in your solutions. No explanation is required.)

(b) Theory Hippo still wants to travel from point (0, 0) to point (32, 43). But he learns there are Bottomless Pits of Utter Annihilation at points (11, 11) and (21, 17) through which paths cannot pass. How many paths are there for Theory Hippo to follow?

Hint: Ignoring the pits, let $S$ be the set of paths from (0, 0) to (32, 43) that go through (11,11), and let $T$ be the ones that go through (21,17).
Problem 13 (Countable Set of Reals) (4 points).
A real number is called quadratic when it is a root of a degree two polynomial with integer coefficients. Explain why there are only countably many quadratic reals.
Problem 14 (Uncountable Stationary Distributions) (4 points).
Explain why there are an uncountable number of stationary distributions for the following random walk graph.

![Random walk graph diagram]