Staff Solutions to In-Class Problems Week 14, Mon.

Problem 1.
A gambler is placing $1 bets on the “1st dozen” in roulette. This bet wins when a number from one to twelve comes in, and then the gambler gets his $1 back plus $2 more. Recall that there are 38 numbers on the roulette wheel.

The gambler’s initial stake in $n$ and his target is $T$. He will keep betting until he runs out of money (“goes broke”) or reaches his target. Let $w_n$ be the probability of the gambler winning, that is, reaching target $T$ before going broke.

(a) Write a linear recurrence for $w_n$; you need not solve the recurrence.

Solution. The probability of winning a bet is $12/38$. Thus, by the Law of Total Probability 16.5.3,

$$w_n = \Pr[\text{win with } n \text{ start } | \text{ won 1st bet}] \cdot \Pr[\text{won 1st bet}] + \Pr[\text{win with } n \text{ start } | \text{ lost 1st bet}] \cdot \Pr[\text{lost 1st bet}]$$

$$= \Pr[\text{win with } n + 2 \text{ start}] \cdot \Pr[\text{won 1st bet}] + \Pr[\text{win with } n - 1 \text{ start}] \cdot \Pr[\text{lost 1st bet}],$$

so

$$w_n = \frac{12}{38} w_{n+2} + \frac{26}{38} w_{n-1}.$$

Letting $m := n + 2$ we get

$$w_m = \frac{38}{12} w_{m-2} - \frac{26}{12} w_{m-3}.$$

As boundary conditions, we have

$$w_0 = 0, w_T = 1.$$

(b) Let $e_n$ be the expected number of bets until the game ends. Write a linear recurrence for $e_n$; you need not solve the recurrence.

Solution. By the Law of Total Expectation, Theorem 17.4.7,

$$e_n = (1 + \text{Ex[#bets with } n \text{ start } | \text{ won 1st bet}]) \cdot \Pr[\text{won 1st bet}] + (1 + \text{Ex[#bets with } n \text{ start } | \text{ lost 1st bet}]) \cdot \Pr[\text{lost 1st bet}]$$

$$= (1 + \text{Ex[#bets with } n + 2 \text{ start}]) \cdot \Pr[\text{won 1st bet}] + (1 + \text{Ex[number of bets starting with } n - 1]) \cdot \Pr[\text{lost 1st bet}],$$

so

$$e_n = (e_{n+2} + 1) \frac{12}{38} + (1 + e_{n-1}) \frac{26}{38}.$$
Letting \( m := n + 2 \) we get
\[
e_m = \frac{38}{12} e_{m-2} - \frac{26}{12} e_{m-3} - \frac{38}{12}
\]
As boundary conditions, we have
\[
e_0 = e_T = 1.
\]

**Problem 2.**
In a gambler’s ruin scenario, the gambler makes independent $1 bets, where the probability of winning a bet is \( p \) and of losing is \( q := 1 - p \). The gambler keeps betting until he goes broke or reaches a target of \( T \) dollars.

Suppose \( T = \infty \), that is, the gambler keeps playing until he goes broke. Let \( r \) be the probability that starting with \( n > 0 \) dollars, the gambler’s stake ever gets reduced to \( n - 1 \) dollars.

(a) Explain why
\[
r = q + pr^2.
\]

**Solution.** By Total Probability
\[
r = \Pr[\text{ever down } \$1 \mid \text{lose the first bet}] \Pr[\text{lose the first bet}]
+ \Pr[\text{ever down } \$1 \mid \text{win the first bet}] \Pr[\text{win the first bet}]
= q + p \Pr[\text{ever down } \$1 \mid \text{win the first bet}]
\]
But
\[
\Pr[\text{ever down } \$1 \mid \text{win the first bet}]
= \Pr[\text{ever down } \$2]
= \Pr[\text{down another } \$1 \mid \text{already down } \$1] \cdot \Pr[\text{ever down } \$1]
= r \cdot r = r^2.
\]

(b) Conclude that if \( p \leq 1/2 \), then \( r = 1 \).

**Solution.** \( pr^2 - r + q \) has roots \( q/p \) and 1. So \( r = 1 \) or \( r = q/p \). But \( r \leq 1 \), which implies \( r = 1 \) when \( q/p \geq 1 \), that is, when \( p \leq 1/2 \).

In fact \( r = q/p \) when \( q/p < 1 \), namely, when \( p > 1/2 \), but this requires an additional argument that we omit.

(c) Prove that even in a fair game, the gambler is sure to get ruined *no matter how much money he starts with!*

**Solution.** The proof is by induction with hypothesis
\[
P(n) := \Pr[\text{ruin starting with } \$n] = 1.
\]

**base case** \( n = 0 \): If the stake is zero, the gambler is ruined at the start, so \( P(0) \) is true by definition.
inductive step: If the gambler’s initial stake is \(n\), the gambler will be ruined iff his stake gets reduced to \(n-1\) and he gets ruined after that. But by part (b), with probability 1 the gambler’s stake will be reduced to \(n-1\), and by induction hypothesis, he will then be ruined also with probability 1. Since the intersection of probability 1 events has probability 1, \(P(n)\) holds.

We conclude by induction that \(\forall n. P(n)\), as claimed.

\((d)\) Let \(t\) be the expected time for the gambler’s stake to go down by 1 dollar. Verify that

\[
t = q + p(1 + 2t).
\]

Conclude that starting with a 1 dollar stake in a fair game, the gambler can expect to play forever!

**Solution.** By Total Expectation

\[
t = \text{Ex}[\text{#steps to be down } \$1 \mid \text{lose the first bet}] \Pr[\text{lose the first bet}] + \\
\text{Ex}[\text{#steps to be down } \$1 \mid \text{win the first bet}] \Pr[\text{win the first bet}] \\
= q + p \text{Ex}[\text{#steps to be down } \$1 \mid \text{win the first bet}].
\]

But

\[
\text{Ex}[\text{#steps to be down } \$1 \mid \text{win the first bet}] \\
= 1 + \text{Ex}[\text{#steps to be down } \$2] \\
= 1 + \text{Ex}[\text{#steps to be down the first } \$1] + \text{Ex}[\text{#steps to be down another } \$1] \\
= 1 + 2t.
\]

This implies the required formula \(t = q + p(1 + 2t)\). If \(p = 1/2\) we conclude that \(t = 1 + t\), which means \(t\) must be infinite.

**Problem 3.**

A gambler bets \$10 on “red” at a roulette table (the odds of red are 18/38, slightly less than even) to win \$10. If he wins, he gets back twice the amount of his bet and he quits. Otherwise, he doubles his previous bet and continues.

**(a)** What is the expected number of bets the gambler makes before he wins?

**Solution.** This is mean time to failure, with failure being a red number coming up. So the expected time (number of bets) is

\[
\frac{1}{18/38} = 2 \frac{1}{9}.
\]

**(b)** What is his probability of winning?

**Solution.** He is certain to win, since \(\Pr[> \ k \text{ bets}] = (20/38)^k\) which goes to zero as \(k\) goes to infinity. More fully,

\[
\Pr[\text{win}] \geq \Pr[\text{win in } \leq k \text{ bets}] = 1 - \Pr[> k \text{ bets}]
\]

and this last expression goes to 1 as \(k\) goes to infinity.
(e) What is his expected final profit (amount won minus amount lost)?

**Solution.** His final profit is always $10 whenever he finally wins, and he is certain to win, so $10 is also his expected final profit.

(d) The fact that the gambler’s expected profit is positive, despite the fact that the game is biased against him, is known as the *St. Petersburg paradox*. The paradox arises from an unrealistic, implicit assumption about the gambler’s money. Explain.

*Hint:* What is the expected size of his last bet?

**Solution.** It sounds plausible that, since his expected number of bets is less than three, the expected size of his bet would be less than the size of his third bet, that is, $10 \cdot 2^2 = 40$ dollars. But this is a sloppy—and wrong—argument. What it overlooks is that later bets, though progressively less likely, grow tremendously, and so contribute heavily to the expected bet size.

To get the answer, we go back to the definition of expected bets. Let $B$ be the size of his last bet in dollars. Now if he wins his $10$ final profit on the $k$th bet, then $B = 10 \cdot 2^{k-1}$, so

$$
\Pr[B = 10 \cdot 2^{k-1}] = (20/38)^{k-1}(18/38)
$$

So

$$
\text{Ex}[B] = \sum_{k \in \mathbb{N}^+} 10 \cdot 2^{k-1} \left(\frac{20}{38}\right)^{k-1} \left(\frac{18}{38}\right)
$$

$$
= 10 \left(\frac{18}{38}\right) \sum_{k \in \mathbb{N}} 2^k \left(\frac{20}{38}\right)^k
$$

$$
> \sum_{k \in \mathbb{N}} \left(1 + \frac{1}{19}\right)^k = \infty
$$

so the gambler has to have an infinite bank account to win $10 with certainty.

**STAFF NOTE:** Maybe ask to verify that his expected gain is negative if he only has a finite number, $n$, of dollars to bet.

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**Problem 4.**

Let $R$ be a positive integer valued random variable such that

$$
\text{PDF}_R(n) = \frac{1}{cn^3},
$$

where

$$
c := \sum_{n=1}^{\infty} \frac{1}{n^3}.
$$

(a) Prove that $\text{Ex}[R]$ is finite.
Solution.

\[
\mathbb{E}[R] := \sum_{n \in \mathbb{N}^+} n \cdot \frac{1}{cn^3} = \sum_{n \in \mathbb{N}^+} \frac{1}{cn^2} < 1 + \int_1^\infty \frac{1}{cx^2} \, dx = 1 + \frac{1}{c} < \infty.
\]

\[\square\]

(b) Prove that \(\text{Var}[R]\) is infinite.

Solution. Since

\[
\text{Var}[R] = \mathbb{E}[R^2] - \mathbb{E}^2[R],
\]

and \(\mathbb{E}^2[R] < \infty\) by part (a), we need only show that \(\mathbb{E}[R^2] = \infty\). But

\[
\mathbb{E}[R^2] := \sum_{n \in \mathbb{N}^+} n^2 \cdot \frac{1}{cn^3} = \sum_{n \in \mathbb{N}^+} \frac{1}{cn} = \frac{1}{c} \cdot \lim_{n \to \infty} H_n = \infty.
\]

\[\square\]

A joking way to phrase the point of this example is “The square root of infinity may be finite.” Namely, let \(T := R^2\). Then the solution to part (b) implies that \(\mathbb{E}[T] = \infty\) while \(\mathbb{E}[\sqrt{T}] < \infty\) by (a).

Problem 5.

You have a biased coin with nonzero probability \(p < 1\) of tossing a Head. You toss until a Head comes up and record the number, \(k\), of Tails that preceded this first Head. Then you keep tossing until you get another run of tails of nearly the same length, namely, of length \(\min\{k - 10, 0\}\). Prove that the expected number of Heads you toss is infinite.

Solution. Let the random variable \(T\) be the length of your initial run of tails. If \(T = k\), then the expected number of Heads tossed until getting another run of Tails of length at least \(k_{10} := \min\{k - 10, 0\}\) will be the mean time to failure, where “failing” means tossing \(k_{10}\) Tails. Since the probability of failure is \(q^{k_{10}}\), where \(q := 1 - p\), this mean time is \(1/q^{k_{10}}\). Letting \(H\) be the number of Heads tossed, we have

\[
\mathbb{E}[H] = \sum_{k \in \mathbb{N}} \mathbb{E}[H \mid T = k] \cdot \Pr[T = k]
= \sum_{k \in \mathbb{N}} \frac{1}{q^{k_{10}}} \cdot q^k p
= \text{constant} + \sum_{k \geq 10} \frac{1}{q^{k-10}} \cdot q^k p
= \text{constant} + p \sum_{k \geq 10} q^{10} = \infty.
\]

\[\square\]