Staff Solutions to In-Class Problems Week 12, Wed.

**STAFF NOTE:** Random Variables, Distributions, Independence

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Guess the Bigger Number Game

**Team 1:**
- Write different integers between 0 and 7 on two pieces of paper.
- Put the papers face down on a table.

**Team 2:**
- Turn over one paper and look at the number on it.
- Either stick with this number or switch to the unseen other number.

Team 2 wins if it chooses the larger number; else, Team 1 wins.

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**Problem 1.**
The analysis in section 17.3.3 implies that Team 2 has a strategy that wins 4/7 of the time no matter how Team 1 plays. Can Team 2 do better? The answer is “no,” because Team 1 has a strategy that guarantees that it wins at least 3/7 of the time, no matter how Team 2 plays. Describe such a strategy for Team 1 and explain why it works.

**Solution.** **STAFF NOTE:** Warn against assuming that Team 2 plays in any particular way such as using thresholds.

To speed things up if need be, explain what the strategy is, and ask for verification that $\Pr[\text{win}] \geq 3/7$ no matter how Team 2 plays.

Team 1 should randomly choose a number $Z \in \{0, \ldots, 6\}$ and write $Z$ and $Z + 1$ on the pieces of paper with all numbers equally likely. Then place the paper with $Z$ on it to the left or right with equal probability.

To see why this works, let $N$ be the number on the paper that Team 2 turns over, and let OK be the event that $N \in [1, 6]$. So given event OK, that is, given that $1 \leq N \leq 6$, Team 1’s strategy ensures that half the time $N$ is the higher number and half the time $N$ is the lower number. So given event OK, the probability that Team 1 wins is exactly $1/2$ no matter how Team 2 chooses to play (stick or switch).

Now we claim that

$$\Pr[\text{OK}] = \frac{6}{7},$$

(1)

which implies (by the Law of Total Probability) that the probability that Team 1 wins is indeed at least $(1/2)(6/7) = 3/7$.

To prove $\Pr[\text{OK}] = 6/7$, we can follow the four step method. (Note that we couldn’t apply this method to model the behavior of Team 2, since we don’t know how they may play, and so we can’t let our analysis depend on what they do.)

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The first level of the probability tree for this game will describe the value of $Z$: there are seven branches from the root with equal probability going to first level nodes corresponding to the seven possible values of $Z$. The second level of the tree describes the choice of the number, $N$: each of the seven first-level nodes has two branches with equal probability, one branch for the case that $N = Z$ and the other for the case that $N = Z + 1$. So there are 14 outcome (leaf) nodes at the second level of the tree, each with probability $1/14$.

<table>
<thead>
<tr>
<th>Value of Z</th>
<th>Value of N</th>
<th>$N$ in ${1, \ldots, 6}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0$</td>
<td>$N = Z$</td>
<td>NOT OK</td>
</tr>
<tr>
<td>$1/7$</td>
<td>$N = Z + 1$</td>
<td>OK</td>
</tr>
<tr>
<td>$1/7$</td>
<td>$N = Z$</td>
<td>OK</td>
</tr>
<tr>
<td>$1/7$</td>
<td>$N = Z$</td>
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<td>$N = Z + 1$</td>
<td>OK</td>
</tr>
</tbody>
</table>

Now only two outcomes are not OK, namely, when $Z = 6$ and $N = 7$, and when $Z = 0$ and $N = 0$. Each of the other twelve outcomes is OK, and since each has probability $1/14$, we conclude that $\Pr[\text{OK}] = 12/14 = 6/7$, as claimed. ■

**Problem 2.** (a) Prove that if $A$ and $B$ are independent events, then so are $A$ and $\overline{B}$.

**Solution.** Proof.

\[
\Pr[A \cap \overline{B}] = \Pr[A] - \Pr[A \cap B]
\quad \text{(difference rule)}
\]
\[
= \Pr[A] - \Pr[A] \cdot \Pr[B]
\quad \text{(independence of $A$ and $B$)}
\]
\[
= \Pr[A] \cdot (1 - \Pr[B])
\]
\[
= \Pr[A] \cdot \Pr[\overline{B}]
\quad \text{(complement rule)}.
\]

(b) Let $I_A$ and $I_B$ be the indicator variables for events $A$ and $B$. Prove that $I_A$ and $I_B$ are independent iff $A$ and $B$ are independent.

**Hint:** For any event, $E$, let $E^1 := E$ and $E^0 := \overline{E}$. So the event $[I_E = a]$ is the same as $E^a$.

**Solution.** Proof. By part (a) and the fact that $\overline{\overline{E}} = E$, the following propositions are equivalent:

- $A$ and $B$ are independent,
- $\exists a, b \in \{0, 1\}$. $[A^a$ and $B^b$ are independent],
• \( \forall a, b \in \{0, 1\}. \) \([A^a \text{ and } B^b \text{ are independent}].\)

Therefore, the following propositions are equivalent as well:

• \( I_A \text{ and } I_B \text{ are independent}, \)
• \( \forall a, b \in \{0, 1\}. \) \([I_A = a] \text{ and } [I_A = b] \text{ are independent events} \)—by definition of independence for random variables,
• \( \forall a, b \in \{0, 1\}. \) \([A^a \text{ and } B^b \text{ are independent}]. \)
• \( A \text{ and } B \text{ are independent} \)—(by part (a)).

**Problem 3.**
Suppose \( R, S, \) and \( T \) are mutually independent random variables on the same probability space with uniform distribution on the range \([1, n]\).

**STAFF NOTE:** That is,

\[
\Pr[U = i] = \frac{1}{n} \quad \text{for } U = R, S, T \text{ and } i \in [1, n].
\]

Let \( M = \max\{R, S, T\}. \)

(a) What is \( \Pr[M \leq k] \) for \( k \in [1, n] \)?

**Solution.**

\[
\left(\frac{k}{n}\right)^3.
\]

To prove this, note that \( M \leq k \) iff \( R \leq k \) AND \( S \leq k \) AND \( T \leq k. \) Since \( R \) is uniform, \( \Pr[R \leq k] = k/n, \) and likewise for \( S \) and \( T, \) and by mutual independence,

\[
\Pr[R \leq k \text{ AND } S \leq k \text{ AND } T \leq k] = \Pr[R \leq k] \cdot \Pr[S \leq k] \cdot \Pr[T \leq k].
\]

(b) Write a formula for PDF\(_M(k)\).

**STAFF NOTE:** *Hint:* \( M = k \) iff \( M \leq k \) AND NOT\((M \leq k - 1).\)

**Solution.**

\[
\text{PDF}_M(1) = \left(\frac{1}{n}\right)^3
\]

\[
\text{PDF}_M(k) = \left(\frac{k}{n}\right)^3 - \left(\frac{k-1}{n}\right)^3 \quad \text{for } k \in (1, n]. \tag{2}
\]

This follows because

\[
\Pr[M \leq k] = \Pr[M = k] + \Pr[M \leq k - 1]
\]

by the Disjoint Sum Rule, and so

\[
\Pr[M = k] = \Pr[M \leq k] - \Pr[M \leq k - 1]
\]

which equals (2) by part (a).
Problem 4.
Suppose you have a biased coin that has probability $p$ of flipping heads. Let $J$ be the number of heads in $n$ independent coin flips. So $J$ has the general binomial distribution:

$$\text{PDF}_J(k) = \binom{n}{k} p^k q^{n-k}$$

where $q := 1 - p$.

(a) Show that

$$\text{PDF}_J(k - 1) < \text{PDF}_J(k) \quad \text{for } k < np + p.$$

$$\text{PDF}_J(k - 1) > \text{PDF}_J(k) \quad \text{for } k > np + p.$$ 

Solution. Consider the ratio of the probability of $k$ heads over the probability of $k - 1$ heads.

$$\frac{\text{PDF}_J(k)}{\text{PDF}_J(k - 1)} = \frac{\binom{n}{k} p^k q^{n-k}}{\binom{n}{k-1} p^{k-1} q^{n-k+1}} = \frac{k! (n-k)! p}{(k-1)! (n-k+1)! q} = \frac{(n - k + 1) p}{k q}$$

This fraction is greater than 1 precisely when $(n - k + 1) p > kq = k(1 - p)$, that is when $k < np + p$. So for $k < np + p$, the probability of $k$ heads increases as $k$ increases, and for $k > np + p$, the probability decreases as $k$ increases.

(b) Conclude that the maximum value of $\text{PDF}_J$ is asymptotically equal to

$$\frac{1}{\sqrt{2\pi npq}}.$$

Hint: For the asymptotic estimate, it’s ok to assume that $np$ is an integer, so by part (a), the maximum value is $\text{PDF}_J(np)$. Use Stirling’s formula (13.25):

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$
Solution.

\[ \text{PDF}_J(np) := \binom{n}{np} p^{np} q^{n-np} \]
\[ = \frac{n!}{(np)! (nq)!} p^{np} q^{nq} \]
\[ \sim \left( \frac{(np)^{np}}{e^{np}} \sqrt{2\pi np} \right) \left( \frac{(nq)^{nq}}{e^{nq}} \sqrt{2\pi nq} \right) p^{np} q^{nq} \]
\[ = \frac{n^n}{e^n \sqrt{2\pi n}} \frac{p^{np} q^{nq}}{e^{np+eq q}} \sqrt{2\pi np} \sqrt{2\pi nq} \]
\[ = \frac{n^n}{e^n \sqrt{2\pi np} \sqrt{2\pi nq}} \frac{p{np} q^{nq}}{\sqrt{2\pi np} \sqrt{2\pi nq}} \]
\[ = \frac{1}{\sqrt{2\pi npq}} \]

Problem 5.
Let \( R \) and \( S \) be independent random variables on the same probability space with the same finite range, \( V \). Suppose \( R \) is uniform — that is,
\[ \Pr[R = v] = \frac{1}{|V|} \]
for all \( v \in V \). Then
\[ \text{the probability that } R = S \text{ is the same as the probability that } R \text{ takes whatever value } S \text{ happens to have and therefore} \]
\[ \Pr[R = S] = \frac{1}{|V|}. \] (3)

The argument above is actually OK, but may seem too informal to be completely convincing. Give a careful proof of this claim.

*Hint:* Use Total Probability on \( \Pr[R = S | S = v] \).

Solution. Proof. STAFF NOTE: If students get stuck, give the first line as a hint. If need be add hint that
\[ [R = S] \cap [S = v] = [R = v] \cap [S = v]. \]
\[
\Pr[R = S] = \sum_{v \in V} \Pr[R = S | S = v] \cdot \Pr[S = v] \\
= \sum_{v \in V} \Pr[R = v | S = v] \cdot \Pr[S = v] \\
= \sum_{v \in V} \Pr[R = v] \cdot \Pr[S = v] \\
= \sum_{v \in V} \frac{1}{|V|} \cdot \Pr[S = v] \\
= \frac{1}{|V|} \cdot \sum_{v \in V} \Pr[S = v] \\
= \frac{1}{|V|} \cdot 1 = \frac{1}{|V|}. 
\]

This proves (3).