Staff Solutions to In-Class Problems Week 11, Mon.

Problem 1.
We are interested in generating functions for the number of different ways to compose a bag of \( n \) donuts subject to various restrictions. For each of the restrictions in (a)-(e) below, find a closed form for the corresponding generating function.

(a) All the donuts are chocolate and there are at least 3.

Solution. There are no ways to select 0, 1, or 2 donuts, and one way to select \( n \) chocolate donuts for each \( n > 2 \), so the generating function is

\[
x^3 + x^4 + x^5 + \cdots = x^3 \left( 1 + x + x^2 + \cdots \right) = \frac{x^3}{1-x}
\]

(b) All the donuts are glazed and there are at most 2.

Solution. There is one way to select 0, 1, or 2 glazed donuts, and no ways to select \( n \) donuts for each \( n > 2 \), so the generating function is

\[1 + x + x^2.\]

(c) All the donuts are coconut and there are exactly 2 or there are none.

Solution.

\[1 + x^2\]

(d) All the donuts are plain and their number is a multiple of 4.

Solution. The generating function is

\[
1 + x^4 + x^8 + \cdots + x^4n + \cdots = \sum_{i=0}^{\infty} (x^4)^n = \frac{1}{1-x^4}
\]

(e) The donuts must be chocolate, glazed, coconut, or plain with the numbers of each flavor subject to the constraints above.
Solution. By the Convolution Rule, the generating function for selecting donuts with these constraints is the product of the preceding generating functions:

$$\frac{x^3}{1-x}(1+x+x^2)(1+x^2) \frac{1}{1-x^4} = \frac{x^3(1+x+x^2)(1+x^2)}{(1-x)^2(1+x)(1+x^2)}$$

$$= \frac{x^3}{(1-x)^2(1+x)}$$

\[\square\]

(f) Find a closed form for the number of ways to select \(n\) donuts subject to the constraints of the previous part.

Solution. Let

$$G(x) := \frac{1+x+x^2}{(1-x)^2(1+x)},$$

so the generating function for donut selections is \(x^3G(x)\). By partial fractions

$$\frac{1+x+x^2}{(1-x)^2(1+x)} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1+x}$$

for some constants, \(A, B, C\). We know that the coefficient of \(x^n\) in the series for \((1-x)^2\) is, by the Convolution Rule, the number of ways to select \(n\) items of two different kinds, namely, \(\binom{n+1}{1} = n + 1\), so we conclude that the \(n\)th coefficient in the series for \(G(x)\) is

$$A + B(n + 1) + C(-1)^n.$$  \hspace{1cm} (2)

To find \(A, B, C\), we multiply both sides of (1) by the denominator \((1-x)^2(1+x)\) to obtain

$$1 + x + x^2 = A(1-x)(1+x) + B(1+x) + C(1-x)^2.$$  \hspace{1cm} (3)

Letting \(x = 1\) in (3), we conclude that \(3 = 2B\), so \(B = 3/2\). Then, letting \(x = -1\), we conclude \((-1)^2 = C2^2\), so \(C = 1/4\). Finally, letting \(x = 0\), we have

$$1 = A + B + C = A + \frac{3}{2} + \frac{1}{4}.$$  

so \(A = -3/4\). Then from (2), we conclude that the \(n\)th coefficient in the series for \(G(x)\) is

$$-\frac{3}{4} + \frac{3(n + 1)}{2} + \frac{(-1)^n}{4} = \frac{6n + 3 + (-1)^n}{4}.$$  

So the \(n\)th coefficient in the series for the generating function, \(x^3G(x)\), for donut selections is zero for \(n < 3\), and, for \(n \geq 3\), is the \((n-3)\)rd coefficient of \(G\), namely,

$$\frac{6n - 15 + (-1)^{n-1}}{4}.$$  \hspace{1cm} \[\square\]

Problem 2.
Miss McGillicuddy never goes outside without a collection of pets. In particular:
She brings a positive number of songbirds, which always come in pairs.

- She may or may not bring her alligator, Freddy.

- She brings at least 2 cats.

- She brings two or more chihuahuas and labradors leashed together in a line.

Let $P_n$ denote the number of different collections of $n$ pets that can accompany her, where we regard chihuahuas and labradors leashed up in different orders as different collections, even if there are the same number chihuahuas and labradors leashed in the line.

For example, $P_6 = 4$ since there are 4 possible collections of 6 pets:

- 2 songbirds, 2 cats, 2 chihuahuas leashed in line
- 2 songbirds, 2 cats, 2 labradors leashed in line
- 2 songbirds, 2 cats, a labrador leashed behind a chihuahua
- 2 songbirds, 2 cats, a chihuahua leashed behind a labrador

And $P_7 = 16$ since there are 16 possible collections of 7 pets:

- 2 songbirds, 3 cats, 2 chihuahuas leashed in line
- 2 songbirds, 3 cats, 2 labradors leashed in line
- 2 songbirds, 3 cats, a labrador leashed behind a chihuahua
- 2 songbirds, 3 cats, a chihuahua leashed behind a labrador
- 4 collections consisting of 2 songbirds, 2 cats, 1 alligator, and a line of 2 dogs
- 8 collections consisting of 2 songbirds, 2 cats, and a line of 3 dogs.

(a) Let

$$P(x) := P_0 + P_1x + P_2x^2 + P_3x^3 + \cdots$$

be the generating function for the number of Miss McGillicuddy’s pet collections. Verify that

$$P(x) = \frac{4x^6}{(1 - x)^2(1 - 2x)}.$$

**Solution.**

$$P(x) = \left(\frac{x^2 + x^4 + x^6 + x^8 + \cdots}{1 - x^2}\right) \cdot \left(\frac{1 + x}{1 - x}\right) \cdot \left(\frac{x^2 + x^3 + x^4 + \cdots}{1 - x}\right) \cdot \left(\frac{2^2x^2 + 2^3x^3 + 2^4x^4 + \cdots}{1 - 2x}\right)$$

$$= \frac{x^2}{1 - x^2} \cdot (1 + x) \cdot \frac{x^2}{1 - x} \cdot \frac{4x^2}{1 - 2x}$$

$$= \frac{x^2}{(1 - x)(1 + x)} \cdot (1 + x) \cdot \frac{x^2}{1 - x} \cdot \frac{4x^2}{1 - 2x}$$

$$= \frac{4x^6}{(1 - x)^2(1 - 2x)}.$$  

(b) Find a simple formula for $P_n$.  

Solution. $P_n$ is the coefficient of $x^n$ in the power series for $4x^6/(1-x)^2(1-2x)$, which means it is 4 times the coefficient of $x^{n-6}$ in the series for $1/(1-x)^2(1-2x)$ when $n \geq 6$, and $P_n = 0$ for $n < 6$.

But we can express $1/(1-x)^2(1-2x)$ using partial fractions as

$$\frac{1}{(1-x)^2(1-2x)} = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1-2x}$$

for some constants $A, B, C$, so $P_n$ will be 4 times the sum of the coefficients of $x^{n-6}$ in each of $A/(1-x)$, $B/(1-x)^2$, and $C/(1-2x)$, namely

$$P_n = 4(A + B \left(\frac{n-5}{1}\right) + C 2^{n-6}) = 4A + 4B(n-5) + C 2^{n-4}. \tag{5}$$

So we need only find the values of $A, B, C$. But multiplying both sides of (4) by the lefthand denominator $(1-x)^2(1-2x)$ yields

$$1 = A(1-x)(1-2x) + B(1-2x) + C(1-x)^2. \tag{6}$$

Now letting $x = 1$ in (6) gives $B = -1$. Similarly, letting $x = 1/2$ gives $C = 4$. Finally, letting $x = 0$ gives $A + B + C = 1$ and so $A = -2$. Substituting these values into (5) finally gives

$$P_n = 4(-2) - 4(n-5) + 4(2^{n-4}) = 2^{n-2} - 4n + 12.$$  

Problem 3.

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$. Then it’s easy to check that

$$a_n = \frac{A^{(n)}(0)}{n!},$$

where $A^{(n)}$ is the $n$th derivative of $A$. Use this fact (which you may assume) instead of the Convolution Counting Principle, to prove that

$$\frac{1}{(1-x)^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n.$$  

Solution.

$$\frac{d}{dx} \frac{1}{(1-x)^k} = k (1-x)^{-(k+1)}.$$  

$$\frac{d^2}{(dx)^2} \frac{1}{(1-x)^k} = k (1-x)^{-(k+1)}.$$  

$$\frac{d^3}{(dx)^3} \frac{1}{(1-x)^k} = (k+1)k (1-x)^{-(k+2)}.$$  

$$\frac{d^n}{(dx)^n} \frac{1}{(1-x)^k} = (k+n-1) \cdots (k+2)(k+1)k (1-x)^{-(k+n)}.$$  

$$\vdots$$
Now suppose \((1 - x)^{-k} = A(x)\). Then we have

\[
a_n = \frac{A^{(n)}(0)}{n!} = \frac{(k + n - 1)(k + n - 2) \cdots (k + 2)(k + 1)k (1 - 0)^{-(k + n)}}{n!} \\
= \frac{(n + k - 1)!}{(k - 1)!} \cdot \frac{1}{n!} \\
= \frac{(n + k - 1)!}{(k - 1)!n!} \\
= \binom{n + k - 1}{k - 1}
\]

So if we didn’t already know the Bookkeeper Rule, we could have proved it from this calculation and the Convolution Rule for generating functions.

**Appendix**

Let \([x^n]F(x)\) denote the coefficient of \(x^n\) in the power series for \(F(x)\). Then,

\[
[x^n] \left( \frac{1}{(1 - ax)^k} \right) = \binom{n + k - 1}{n} \alpha^n.
\]

**Partial Fractions**

Here’s a particular case of the Partial Fraction Rule that should be enough to illustrate the general Rule. Let

\[
r(x) := \frac{p(x)}{(1 - ax)^2(1 - bx)(1 - cx)^3}
\]

where \(\alpha, \beta, \gamma\) are distinct nonzero, complex numbers, and \(p(x)\) is a polynomial of degree less than the denominator, namely, less than 6. Then there are unique numbers \(a_1, a_2, b, c_1, c_2, c_3 \in \mathbb{C}\) such that

\[
r(x) = \frac{a_1}{1 - \alpha x} + \frac{a_2}{(1 - \alpha x)^2} + \frac{b}{1 - \beta x} + \frac{c_1}{1 - \gamma x} + \frac{c_2}{(1 - \gamma x)^2} + \frac{c_3}{(1 - \gamma x)^3}
\]