Staff Solutions to In-Class Problems Week 10, Wed.

Problem 1.
The Tao of BOOKKEEPER: we seek enlightenment through contemplation of the word BOOKKEEPER.

(a) In how many ways can you arrange the letters in the word POKE?

Solution. There are $4!$ arrangements corresponding to the $4!$ permutations of the set $\{P, O, K, E\}$. ■

(b) In how many ways can you arrange the letters in the word $BO_1O_2K$? Observe that we have subscripted the O’s to make them distinct symbols.

Solution. There are $4!$ arrangements corresponding to the $4!$ permutations of the set $\{B, O_1, O_2, K\}$. ■

(c) Suppose we map arrangements of the letters in $BO_1O_2K$ to arrangements of the letters in BOOK by erasing the subscripts. Indicate with arrows how the arrangements on the left are mapped to the arrangements on the right.

\[
\begin{align*}
O_2BO_1K & \quad \Rightarrow \quad BOOK \\
KO_2BO_1 & \\
O_1BO_2K & \quad \Rightarrow \quad OBOK \\
KO_1BO_2 & \quad \Rightarrow \quad KOBO \\
BO_1O_2K & \\
BO_2O_1K & \\
\ldots & \\
\end{align*}
\]

(d) What kind of mapping is this, young grasshopper?

Solution. 2-to-1 ■

(e) In light of the Division Rule, how many arrangements are there of BOOK?

Solution. $4!/2$ ■

(f) Very good, young master! How many arrangements are there of the letters in $KE_1E_2PE_3R$?

Solution. $6!$ ■

(g) Suppose we map each arrangement of $KE_1E_2PE_3R$ to an arrangement of KEEPER by erasing subscripts. List all the different arrangements of $KE_1E_2PE_3R$ that are mapped to REPEEK in this way.

Solution. $RE_1PE_2E_3K, RE_1PE_3E_2K, RE_2PE_1E_3K, RE_2PE_3E_1K, RE_3PE_1E_2K, RE_3PE_2E_1K$ ■

(h) What kind of mapping is this?
(i) So how many arrangements are there of the letters in $KEEPER$?

Solution. $6!/3!$

Now you are ready to face the $BOOKKEEPER$!

(j) How many arrangements of $BO_1O_2K_1K_2E_1E_2PE_3R$ are there?

Solution. $10!$

(k) How many arrangements of $BOOK_1K_2E_1E_2PE_3R$ are there?

Solution. $10!/2!$

(l) How many arrangements of $BOOKKE_1E_2PE_3R$ are there?

Solution. $10!/(2! \cdot 2!)$

(m) How many arrangements of $BOOKKEEPER$ are there?

Solution.

$$
\binom{10}{1,2,2,3,1,1} := \frac{10!}{1! \cdot 2! \cdot 2! \cdot 3! \cdot 1! \cdot 1!} = \frac{10!}{(2!)^2 \cdot 3!}
$$

Remember well what you have learned: subscripts on, subscripts off. This is the Tao of Bookkeeper.

(n) How many arrangements of $VOODOODOLL$ are there?

Solution.

$$
\binom{10}{1,2,5,2} := \frac{10!}{1! \cdot 2! \cdot 5! \cdot 2!}
$$

(o) How many length 52 sequences of digits contain exactly 17 two’s, 23 fives, and 12 nines?

Solution.

$$
\binom{52}{17,23,12} := \frac{52!}{17! \cdot 23! \cdot 12!}
$$
Problem 2. (a) Use the Multinomial Theorem 14.6.5 to prove that

\[(x_1 + x_2 + \cdots + x_n)^p \equiv x_1^p + x_2^p + \cdots + x_n^p \pmod{p}\]  

(1)

for all primes \(p\). (Do not prove it using Fermat’s “little” Theorem. The point of this problem is to offer an independent proof of Fermat’s theorem.)

*Hint:* Explain why \(\binom{p}{k_1, k_2, \ldots, k_n}\) is divisible by \(p\) if all the \(k_i\)’s are positive integers less than \(p\).

Solution. By the Multinomial Theorem 14.6.5, \((x_1 + x_2 + \cdots + x_n)^p\) is a sum of monomials in \(x_1, \ldots, x_n\) whose coefficients are

\[\binom{p}{k_1, k_2, \ldots, k_n}\]

where the sum of the \(k_i\)’s is \(p\). But if all the \(k_i\)’s are less than \(p\), then none of the denominator terms divides the numerator, \(p\), and so the multinomial coefficient is divisible by \(p\). So the only coefficients not divisible by \(p\) are the coefficients of the terms \(x_i^p\), and all the other terms are \(0 \pmod{p}\).

(b) Explain how (1) immediately proves Fermat’s Little Theorem 8.10.11: \(n^{p-1} \equiv 1 \pmod{p}\) when \(n\) is not a multiple of \(p\).

Solution. Let \(x_1 = x_2 = \cdots x_n = 1\). Then (1) implies \(n^p \equiv n \cdot 1^p = n \pmod{p}\). If \(n\) is not a multiple of \(p\), then we can then cancel \(n\) to get \(n^{p-1} \equiv 1 \pmod{p}\).

Problem 3.

Here are the solutions to the next 7 short answer questions, in no particular order. Indicate the solutions for the questions and briefly explain your answers.

1. \(\frac{n!}{(n-m)!}\)  
2. \(\binom{n+m}{m}\)  
3. \((n-m)!\)  
4. \(m^n\)  
5. \(\binom{n-1+m}{m}\)  
6. \(\binom{n-1+m}{n}\)  
7. \(2^{mn}\)  
8. \(n^m\)

(a) How many length \(m\) words can be formed from an \(n\)-letter alphabet, if no letter is used more than once?

Solution.

\[\frac{n!}{(n-m)!}\]

There are \(n\) choices for the first letter, \(n-1\) choices for the second letter, \(\ldots\) \(n-m+1\) choices for the \(m\)th letter, so by the Generalized Product rule, the number of words is

\[n \cdot (n-1) \cdots (n-m+1)\].

(b) How many length \(m\) words can be formed from an \(n\)-letter alphabet, if letters can be reused?

Solution. \(n^m\) by the Product Rule.

(c) How many binary relations are there from set \(A\) to set \(B\) when \(|A| = m\) and \(|B| = n\)?
Solution.

The graph of a binary relations from $A$ to $B$ is a subset of $A \times B$. There are on $2^{mn}$ such subsets because $|A \times B| = mn$.

(d) How many total injective functions are there from set $A$ to set $B$, where $|A| = m$ and $|B| = n \geq m$?

Solution.

\[ \frac{n!}{(n-m)!} \]

There is a bijection between the injections and the length $m$ sequences of distinct elements of $B$. By the Generalized Product rule, the number of such sequences is

\[ n \cdot (n - 1) \cdots (n - m + 1). \]

(e) How many ways are there to place a total of $m$ distinguishable balls into $n$ distinguishable urns, with some urns possibly empty or with several balls?

Solution.

\[ n^m \]

There is a bijection between a placement of the balls and length $m$ sequence whose $i$th element is the urn where the $i$th ball is placed. So the number of placements is the same as the number of length $m$ sequences of elements from a size-$n$ set.

(f) How many ways are there to place a total of $m$ indistinguishable balls into $n$ distinguishable urns, with some urns possibly empty or with several balls?

Solution.

\[ \binom{n - 1 + m}{m} \]

This is the same as the number of selections of $m$ donuts with $n$ possible flavors, which is the number of bit-sequences with $m$ 0’s and $n - 1$ 1’s.

(g) How many ways are there to put a total of $m$ distinguishable balls into $n$ distinguishable urns with at most one ball in each urn?

Solution.

\[ \frac{n!}{(n-m)!} \]

There is a bijection between a placement of balls and a length $m$ sequence whose $i$th element is the urn containing the $i$th ball. So the number of ball placements is the same as number of length $m$ sequences of distinct elements from a set of $n$ elements.

Problem 4.

Solve the following counting problems. Define an appropriate mapping (bijective or $k$-to-$1$) between a set whose size you know and the set in question.
(a) An independent living group is hosting nine new candidates for membership. Each candidate must be assigned a task: 1 must wash pots, 2 must clean the kitchen, 3 must clean the bathrooms, 1 must clean the common area, and 2 must serve dinner. Write a multinomial coefficient for the number of ways this can be done.

**Solution.** There is a bijection from sequences containing one $P$, two $K$’s, three $B$’s, a $C$, and two $D$’s. In any such sequence, the letter in the $i$th position specifies the task assigned to the $i$th candidate. Therefore, the number of possible assignments is:

\[
\binom{9}{1, 2, 3, 1, 2} := \frac{9!}{1! \cdot 2! \cdot 3! \cdot 1! \cdot 2!}
\]

(b) How many nonnegative integers less than 1,000,000 have exactly one digit equal to 9 and have a sum of digits equal to 17?

**Solution.** We identify the nonnegative integers less than 1,000,000 with the length 6 strings of decimal digits. Then there is a bijection with pairs:

(position of the 9, successive values of other 5 digits)

The sum of the other 5 digits is equal to 8, so the number of ways to choose their values is equal to the number of solutions over the nonnegative integers to

\[x_1 + x_2 + x_3 + x_4 + x_5 = 8,\]

namely, \(\binom{12}{4}\). So by the product rule there are

\[6 \cdot \binom{12}{4}\]

such integers.