Staff Solutions to Problem Set 12

Reading: This pset covers Notes Ch. 18, skipping §§18.3.5–18.3.7 and Ch.??, Additional reading for class May 7–12 is specified in online Tutor Problems 13.

STAFF NOTE: TOPICS 1: Probability, Conditional Probability, Independence

Problem 1.
Outside of their hum-drum duties as 6.042 LAs, Oscar is trying to learn to levitate using only intense concentration and Liz is trying to become the world champion flaming torch juggler. Suppose that Oscar’s probability of success is $1/6$, Liz’s chance of success is $1/4$, and these two events are independent.

(a) If at least one of them succeeds, what is the probability that Oscar learns to levitate?

Solution. Let $L$ be the event that Oscar learns to levitate, and let $F$ be the event that Liz becomes the flaming torch juggler champion. We can work out the desired probability as follows:

$$\Pr\{L \mid (L \cup F)\} = \frac{\Pr\{L \cap (L \cup F)\}}{\Pr\{L \cup F\}} = \frac{\Pr\{L\}}{1 - \Pr\{L \cap F\}} = \frac{1/6}{1 - (1 - 1/6)(1 - 1/4)} = \frac{4}{9}$$

The first step uses the definition of conditional probability. In the second step, we rewrite both the top and bottom of the fraction using set identities. Then we substitute in the given probability and simplify.

(b) If at most one of them succeeds, what is the probability that Liz becomes the world flaming torch juggler champion?

Solution. Define events $L$ and $F$ as before.
\[
\Pr\{F \mid (L \cup \overline{F})\} = \frac{\Pr\{F \cap (L \cup \overline{F})\}}{\Pr\{L \cup \overline{F}\}} \\
= \frac{\Pr\{F \cap L\}}{1 - \Pr\{L \cap F\}} \\
= \frac{(1/4) \cdot (5/6)}{1 - (1/6) \cdot (1/4)} \\
= \frac{5}{23}
\]

(c) If exactly one of them succeeds, what is the probability that it is Oscar?

Solution.

\[
\Pr\{L \mid ((L \cap F) \cup (\overline{L} \cap F))\} = \frac{\Pr\{L \cap \overline{F}\}}{\Pr\{((L \cap F) \cup (\overline{L} \cap F))\}} \\
= \frac{(1/6) \cdot (3/4)}{(1/6) \cdot (3/4) + (5/6) \cdot (1/4)} \\
= \frac{3}{8}
\]

STAFF NOTE: TOPICS 2: Random Variables, Expectation

Problem 2.
Here are seven propositions:

\[
\begin{align*}
x_1 & \lor x_3 & \lor \neg x_7 \\
\neg x_5 & \lor x_6 & \lor x_7 \\
x_2 & \lor \neg x_4 & \lor x_6 \\
\neg x_4 & \lor x_5 & \lor \neg x_7 \\
x_3 & \lor \neg x_5 & \lor \neg x_8 \\
x_9 & \lor \neg x_8 & \lor x_2 \\
\neg x_3 & \lor x_9 & \lor x_4
\end{align*}
\]

Note that:

1. Each proposition is the OR of three terms of the form \(x_i\) or the form \(\neg x_i\).
2. The variables in the three terms in each proposition are all different.

Suppose that we assign true/false values to the variables \(x_1, \ldots, x_9\) independently and with equal probability.
(a) What is the expected number of true propositions?

Solution. Each proposition is true unless all three of its terms are false. Thus, each proposition is true with probability:

$$1 - \left(\frac{1}{2}\right)^3 = \frac{7}{8}$$

Let $T_i$ be an indicator for the event that the $i$-th proposition is true. Then the number of true propositions is $T_1 + \ldots + T_7$ and the expected number is:

$$E[T_1 + \ldots + T_7] = E[T_1] + \ldots + E[T_7]$$
$$= \frac{7}{8} + \ldots + \frac{7}{8}$$
$$= \frac{49}{8} = 6\frac{1}{8}$$

(b) Use your answer to prove that there exists an assignment to the variables that makes all of the propositions true.

Solution. A random variable can not always be less than its expectation, so there must be some assignment such that:

$$T_1 + \ldots T_7 \geq \frac{7}{8}$$

This implies that $T_1 + \ldots + T_7 = 7$ for at least one outcome. This outcome is an assignment to the variables such that all of the propositions are true.

Problem 3.

Each 6.042 final exam will be graded according to a rigorous procedure:

- With probability $\frac{4}{7}$ the exam is graded by a TA, with probability $\frac{2}{7}$ it is graded by a lecturer, and with probability $\frac{1}{7}$, it is accidentally dropped behind the radiator and arbitrarily given a score of 84.

- TAs score an exam by scoring each problem individually and then taking the sum.
  - There are ten true/false questions worth 2 points each. For each, full credit is given with probability $\frac{3}{4}$, and no credit is given with probability $\frac{1}{4}$.
  - There are four questions worth 15 points each. For each, the score is determined by rolling two fair dice, summing the results, and adding 3.
  - The single 20 point question is awarded either 12 or 18 points with equal probability.

- Lecturers score an exam by rolling a fair die twice, multiplying the results, and then adding a “general impression” score.
  - With probability $\frac{4}{10}$, the general impression score is 40.
  - With probability $\frac{3}{10}$, the general impression score is 50.
– With probability $\frac{3}{10}$, the general impression score is 60.

Assume all random choices during the grading process are independent.

(a) What is the expected score on an exam graded by a TA?

**Solution.** Let the random variable $T$ denote the score a TA would give. By linearity of expectation, the expected sum of the problem scores is the sum of the expected problem scores. Therefore, we have:

$$E[T] = 10 \cdot E[T/F \text{ score}] + 4 \cdot E[15pt \text{ prob score}] + E[20pt \text{ prob score}]$$

$$= 10 \cdot \left(\frac{3}{4} \cdot 2 + \frac{1}{4} \cdot 0\right) + 4 \cdot \left(\frac{7}{2} \cdot \frac{2}{2} + \frac{7}{2} \cdot \frac{1}{2} \cdot 18\right)$$

$$= 10 \cdot \frac{3}{2} + 4 \cdot 10 + 15$$

$$= 70$$

(b) What is the expected score on an exam graded by a lecturer?

**Solution.** Now we find the expected value of $L$, the score a lecturer would give. Employing linearity again, we have:

$$E[L] = E[\text{product of dice}] + E[\text{general impression}]$$

$$= \left(\frac{7}{2}\right)^2 + \left(\frac{4}{10} \cdot 40 + \frac{3}{10} \cdot 50 + \frac{3}{10} \cdot 60\right)$$

$$= \frac{49}{4} + 49$$

$$= 61 \frac{1}{4}$$

(c) What is the expected score on a 6.042 final exam?

**Solution.** Let $X$ equal the true exam score. The total expectation law implies:

$$E[X] = \frac{4}{7} \cdot E[T] + \frac{2}{7} \cdot E[L] + \frac{1}{7} \cdot 84$$

$$= \frac{4}{7} \cdot 70 + \frac{2}{7} \cdot \left(\frac{49}{4} + 49\right) + \frac{1}{7} \cdot 84$$

$$= 40 + \frac{7}{2} + 14 + 12$$

$$= 69 \frac{1}{2}$$
Problem 4.
The most common way to build a local area network nowadays is an Ethernet: basically, a single wire to which we can attach any number of computers. The communication protocol is quite simple: anyone who wants to talk to another computer broadcasts a message on the wire, hoping the other computer will hear it. The problem is that if more than one computer broadcasts at once, a collision occurs that garbles all messages we are trying to send. The transmission only works if exactly one machine broadcasts at one time.

Let’s consider a simple example. There are \( n \) machines connected by an ethernet, and each wants to broadcast a message. We can imagine time divided into a sequence of intervals, each of which is long enough for one message broadcast.

Suppose each computer flips an independent coin, and decides to broadcast with probability \( p \).

(a) What is the probability that exactly one message gets through in a given interval? **Hint:** Consider the event \( A_i \) that machine \( i \) transmits but no other does.

**Solution.**

\[
\Pr \{ A_i \} = p(1 - p)^{n-1}. \tag{1}
\]

Thus, since the events \( A_i \) are disjoint, we have

\[
\Pr \left\{ \bigcup A_i \right\} = \sum_{i=1}^{n} \Pr \{ A_i \} = np(1 - p)^{n-1}. \tag{2}
\]

(b) What is the expected time it takes for machine \( i \) to get a message through?

**Solution.** This is just mean time to failure, where machine \( i \) getting a message through is the “failure.” So the answer is \( 1/P \) where \( P := \Pr \{ A_i \} \) is given by (1).

**STAFF NOTE:** The following part should have been included:

What value of \( p \) (as a function of \( n \)) minimizes the expected time for the network to successfully transmit some message? Conclude that the network can be expected to transmit a message after at most three intervals.

**Solution.** We want to maximize the expression (2) as a function of \( p \). Differentiating the above expression with respect to \( p \) and setting to zero gives the equation:

\[
(1 - p)^{n-1} - (n - 1)p(1 - p)^{n-2} = 0
\]

\[
(1 - p) - (n - 1)p = 0
\]

\[
p = \frac{1}{n}
\]

Plugging in the \( 1/n \) for \( p \) in (2), we find the maximum probability that some message gets through successfully is

\[
np(1 - p)^{n-1} = \left( 1 - \frac{1}{n} \right)^{n-1} \sim 1/e,
\]

so the minimized expected time is asymptotically equal to \( e < 3 \).