Final Examination

Your name:__________________________________________________________

Circle the name of your TA:    Aditya    Jay    Jeremy    Ling

There are fourteen (14) problems totaling 100 points. Total time is 3 hours.
Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Please keep your entire answer to a problem on that problem’s page.
No reference material or calculators are allowed except for two two-sided, handwritten crib sheets. There is an Appendix that repeats some material from the class Notes. You may assume without proof any of the results presented in class or the course material. GOOD LUCK!

DO NOT WRITE BELOW THIS LINE

This is where the grades table is supposed to be printed. Compile the paper once more to get it right. See course.sty for details.
Problem 1 (6 points). Simple Graphs

(a) (2 points) Prove that the average degree of a tree is less than 2. Hint: Handshaking.

(b) (4 points) The following parts refer to the graph, $H$, in Figure 1 below.

1. How many isomorphisms are there from $H$ to itself? 

2. What is the largest $k$ such that $H$ remains connected as long as fewer than $k$ edges are deleted? 

3. What is the chromatic number of $H$? 

![Figure 1: The graph, $H$.]
Problem 2 (6 points). **Graph Coloring**

Recall that a *coloring* of a graph is an assignment of a color to each vertex such that no two adjacent vertices have the same color. A *k-coloring* is a coloring that uses at most *k* colors.

**False Claim.** Let *G* be a graph whose vertex degrees are all $\leq k$. If *G* has a vertex of degree strictly less than *k*, then *G* is *k*-colorable.

(a) **(1 point)** Give a counterexample to the False Claim when $k = 2$.

(b) **(3 points)** Underline the exact sentence or part of a sentence where the following proof of the False Claim first goes wrong:

**False proof.** Proof by induction on the number $n$ of vertices:

**Induction hypothesis:**

$P(n) := \text{“Let } G \text{ be an } n\text{-vertex graph whose vertex degrees are all } \leq k. \text{ If } G \text{ also has a vertex of degree strictly less than } k, \text{ then } G \text{ is } k\text{-colorable.”} \quad (1)$

**Base case:** ($n = 1$) $G$ has one vertex, the degree of which is 0. Since $G$ is 1-colorable, $P(1)$ holds.

**Inductive step:**

We may assume $P(n)$. To prove $P(n + 1)$, let $G_{n+1}$ be a graph with $n + 1$ vertices whose vertex degrees are all $k$ or less. Also, suppose $G_{n+1}$ has a vertex, $v$, of degree strictly less than $k$. Now we only need to prove that $G_{n+1}$ is $k$-colorable.

To do this, first remove the vertex $v$ to produce a graph, $G_n$, with $n$ vertices. Let $u$ be a vertex that is adjacent to $v$ in $G_{n+1}$. Removing $v$ reduces the degree of $u$ by 1. So in $G_n$, vertex $u$ has degree strictly less than $k$. Since no edges were added, the vertex degrees of $G_n$ remain $\leq k$. So $G_n$ satisfies the conditions of the induction hypothesis, $P(n)$, and so we conclude that $G_n$ is $k$-colorable.

Now a $k$-coloring of $G_n$ gives a coloring of all the vertices of $G_{n+1}$, except for $v$. Since $v$ has degree less than $k$, there will be fewer than $k$ colors assigned to the nodes adjacent to $v$. So among the $k$ possible colors, there will be a color not used to color these adjacent nodes, and this color can be assigned to $v$ to form a $k$-coloring of $G_{n+1}$. \[\square\]
(c) (2 points) With a slightly strengthened condition, the preceding proof of the False Claim could be revised into a sound proof of the following Claim:

Claim. Let $G$ be a graph whose vertex degrees are all $\leq k$. If (statement inserted from below) has a vertex of degree strictly less than $k$, then $G$ is $k$-colorable.

Circle each of the statements below that could be inserted to make the Claim true.

- $G$ is connected and
- $G$ has no vertex of degree zero and
- $G$ does not contain a complete graph on $k$ vertices and
- every connected component of $G$
- some connected component of $G$
Problem 3 (4 points). Probabilistic Land of Paradox

Recall the land of Paradox where everyone is either a liar who always claims a statement is true iff it is actually false, or else is a truth-teller who always does the opposite. Let \( L \) be the event that a randomly chosen Paradoxian, call him “Bill,” is a liar.

The only way out of Paradox at any time is through the North gate or the South gate, and exactly one of these gates is kept open. Which gate is open is determined by independently flipping a biased coin. Let \( N \) be the event that the way out is through the North gate. Finally, let \( R \) the event that Bill’s responds “true” to the statement \( L \iff \overline{N} \).”

(a) (2 points) Draw a tree diagram that describes a sample space for events \( L \) and \( N \). Conclude that \( R = N \).

(b) (2 points) Suppose \( \Pr \{N\} = 1/4, \Pr \{L\} = 1/3 \). Given that Bill’s response, \( R \), is “true,” what is the probability that Bill is a liar?
Problem 4 (7 points). Counting & Probability
For the following problem parts, you do not need to simplify your answers.

(a) (2 points) Each week Bob purchases exactly 10 packages of meat and 20 packages of vegetables for his fraternity. There are three kinds of meat and five kinds of vegetables he can choose from. How many different collections of meat and vegetables can Bob form?

(b) (2 points) You are playing TextTwist®, and you’ve gotten the letters ADEEHR. The two winning 6-letter words are ADHERE and HEADER. You press ”Twist,” which randomly scrambles the letters. What is the probability that your twist returns one of the winning words?
(c) **3 points** An $A,B,C,D$-labeling assigns letters $A,B,C,D$ to each vertex of a graph such that adjacent vertices have different labels. How many $A,B,C,D$-labelings are there of the above graph?
Problem 5 (6 points). DAG’s

Sauron wants to conquer Middle Earth. This project involves $n$ tasks. Some of these tasks must be completed before others are begun. For example, the task of locating the One Ring must precede the task of seizing the ring. Each task can be completed by a horrible creature called a Ringwraith in exactly one week. A Ringwraith can complete multiple tasks, but can only work on one at a time.

(a) (1 point) Sauron would like to model this scheduling problem as a directed acyclic graph. What should vertices represent? When should there be a directed edge between two vertices?

(b) (2 points) Sauron is trying to describe various features of his scheduling problem using standard terminology. Next to each feature below, write the number of the corresponding term.

<table>
<thead>
<tr>
<th>Standard Terminology</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Positive path relation</td>
<td></td>
</tr>
<tr>
<td>2. Topological sort</td>
<td></td>
</tr>
<tr>
<td>3. Chain</td>
<td></td>
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<tr>
<td>4. Antichain</td>
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<tr>
<td>5. Size of the largest antichain</td>
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<tr>
<td>6. Length of the longest chain</td>
<td></td>
</tr>
<tr>
<td>7. Size of the smallest antichain</td>
<td></td>
</tr>
<tr>
<td>8. Length of the shortest chain</td>
<td></td>
</tr>
</tbody>
</table>

A set of tasks that can be worked on simultaneously.   
A possible order in which all the tasks could be completed, if only one Ringwraith were available.   
The minimum number of weeks required to complete all tasks, if an unlimited number of Ringwraiths were available.
(c) (3 points) Sauron’s chief intelligence officer analyzes the dependencies between tasks and reports that Sauron could conquer Middle Earth in $t$ weeks, if he had enough Ringwraiths. Based on this, Sauron realizes he might have to recruit as many as $n - (t - 1)$ Ringwraiths to complete all $n$ tasks in $t$ weeks. Describe a possible dependency graph on the $n$ tasks that would, in fact, require Sauron to use this many Ringwraiths.
Problem 6 (8 points). Number Theory

(a) (3 points) Prove that $5^{16^8n} \equiv 110^{16^8n} \pmod{245}$ for all positive integers, $n$. 
(b) (2 points) Use Fermat’s Little Theorem to calculate \( \text{rem}(8^{29}, 5) \).

(c) (3 points) Part (b) implies that \( \text{rem}(8^{29}, 5) \neq 1 \). It follows that there must be a mistake in the following proof:

\[
\begin{align*}
1 \neq \text{rem}(8^{29}, 5) \pmod{5} & \quad \text{(by part (b), since } 0 \leq \text{rem}(a, 5) < 5) \\
\equiv 8^{29} \pmod{5} & \quad (a \equiv \text{rem}(a, n) \pmod{n}) \\
\equiv 3^{29} \pmod{5} & \quad (\text{since } 8 \equiv 3 \pmod{5}) \\
\equiv 3^4 \pmod{5} & \quad (\text{since } 29 \equiv 4 \pmod{5}) \\
\equiv 1 \pmod{5} & \quad (\text{by Fermat’s})
\end{align*}
\]

Explain the mistake in the reasoning of this proof.
Problem 7 (12 points). Combinatorics and Counting

(a) (10 points) Here are the solutions to the next 10 counting questions, in no particular order:

\[ n^m \quad m^n \quad \frac{n!}{(n-m)!} \quad \binom{n-1+m}{m} \quad \binom{n-1+m}{n} \quad 2^{mn} \]

1. How many ways are there to put \( m \) indistinguishable balls into \( n \) distinguishable urns?

2. How many ways are there to put \( m \) distinguishable balls into \( n \) distinguishable urns?

3. How many solutions over the nonnegative integers are there to the equation \( x_1 + x_2 + \ldots + x_m = n \)?

4. How many \( \sqrt{n} \times \sqrt{n} \) matrices are there with entries drawn from \( \{1, 2, \ldots, m\} \), if entries can be reused?

5. How many different subsets of the set \( A \times B \) are there, if \( |A| = m \) and \( |B| = n \)?

6. How many total functions are there from set \( A \) to set \( B \), if \( |A| = n \) and \( |B| = m \)?

7. How many \( m \)-letter words can be formed from an \( n \)-letter alphabet, if no letter is used more than once? Assume \( n \geq m \).

8. How many \( m \)-letter words can be formed from an \( n \)-letter alphabet, if letters can be reused?

9. How many relations are there from set \( A \) to set \( B \), where \( |A| = m \) and \( |B| = n \)?

10. How many total injective functions are there from set \( A \) to set \( B \), where \( |A| = m \) and \( |B| = n \geq m \)?
(b) (2 points) Here is a combinatorial proof of an equation giving a closed form for a sum from 0 to \( n \).

There are \( n \) fire hydrants, each of which is to be painted red, green, blue, black, or white. One way to paint them is choose one of the five colors for each hydrant successively. An alternative way to assign colors to the hydrants, is to

- choose a number, \( i \), between 0 and \( n \),
- choose a set, \( S \), of \( i \) hydrants,
- successively paint the hydrants in \( S \) red, green, or blue,
- successively paint the hydrants not in \( S \) black or white.

What is the equation?
Problem 8 (6 points). **Communication Networks**

The notes described the $n$-input grid network and proved it had congestion 2 (see the Appendix). In this problem we consider a network called an $n$-input 2-layer-grid consisting of two $n$-input grids connected as pictured below for $n = 4$.

In general, an $n$-input 2-layer-grid has two layers of switches, with each layer connected like an $n$-input grid. There is also an edge from each switch in the first layer to the corresponding switch in the second layer. The inputs of the 2-layer-grid enter the left side of the first layer, and the $n$ outputs leave the bottom of the second layer.

(a) **(2 points)** What is the latency of an $n$-input 2-layer-grid? ______

(b) **(4 points)** For any given input-output permutation, there is a way to route packets that achieves congestion 1. Describe how to route the packets in this way.
Problem 9 (7 points). **Structural Induction**

The rational functions of a single variable, \( x \), are defined recursively as follows:

**Base cases:**

- The identity function, \( \text{id}(x) ::= x \), and
- any constant function

are rational functions of \( x \).

**Constructor cases:**

If \( f, g \) are rational functions of \( x \), then so are

\[
f + g, \quad f \cdot g, \quad \text{and} \quad \frac{f}{g}.
\]

In this problem you will use this definition of rational functions to prove the following Lemma by structural induction:

**Lemma.** The derivative, \( h' \), of a rational function, \( h(x) \), is also a rational function of \( x \).

The induction hypothesis will be \( P(h) ::= [h' \text{ is a rational function}] \).

(a) **(2 points)** Prove the base cases of the structural induction.

(b) **(5 points)** Prove the constructor cases of the structural induction.
Problem 10 (6 points). Generating Functions for \( k \)th Powers

Let \( g_k \) be the generating function for the \( k \)th powers of the nonnegative integers, that is,

\[
g_k(x) ::= \underbrace{0^k + 1^k x + 2^k x^2 + \cdots + n^k x^n + \cdots}_n,
\]

(where \( 0^0 ::= 1 \) by convention).

In this problem you will prove by induction that \( g_k(x) \) is a rational function of \( x \). You may assume the results of problem 9.

(a) (1 point) Give a simple formula defining a rational function of \( x \) equal to \( g_0(x) \).

(b) (2 points) Write a simple expression for \( g_{k+1} \) in terms of the derivative of \( g_k \).

(c) (3 points) Define

\[
P(k) ::= \text{[}g_k(x) \text{ is a rational function of } x\text{]}.
\]

Prove that \( \forall k \in \mathbb{N}. P(k) \) by induction.

You should carefully state the induction hypothesis, the induction variable, the base case and the induction step. You may assume all the results of Problem 9 and the previous parts of this problem.
Problem 11 (4 points). Stationary Distributions

For which of the following graphs is the uniform distribution over nodes a stationary distribution? The edges are labeled with transition probabilities. Circle all that apply.
Problem 12 (11 points). **Graphs, Logic & Probability**  
Let $G$ be an undirected simple graph with $n > 3$ vertices. Let $E(x, y)$ mean that $G$ has an edge between vertices $x$ and $y$, and let $P(x, y)$ mean that there is a length 2 path in $G$ between $x$ and $y$.

(a) (1 point) Explain why $E(x, y)$ implies $P(x, x)$.

(b) (1 point) Circle the mathematical formula that best expresses the definition of $P(x, y)$.

- $P(x, y) ::= \exists z. E(x, z) \land E(y, z)$
- $P(x, y) ::= x \neq y \land \exists z. E(x, z) \land E(y, z)$
- $P(x, y) ::= \forall z. E(x, z) \lor E(y, z)$
- $P(x, y) ::= \forall z. x \neq y \rightarrow [E(x, z) \lor E(y, z)]$

For the following parts (c)–(e), let $V$ be a fixed set of $n > 3$ vertices, and let $G$ be a graph with these vertices constructed randomly as follows: for all distinct vertices $x, y \in V$, independently include edge $x$—$y$ as an edge of $G$ with probability $p$. In particular, $\Pr \{E(x, y)\} = p$ for all $x \neq y$.

(c) (4 points) For distinct vertices $w, x, y$ and $z$ in $V$, circle the event pairs that are independent.

1. $E(w, x)$ versus $E(x, y)$
2. $(E(w, x) \land E(w, y))$ versus $(E(z, x) \land E(z, y))$
3. $E(x, y)$ versus $P(x, y)$
4. $P(w, x)$ versus $P(x, y)$
5. $P(w, x)$ versus $P(y, z)$
(d) (3 points) Write a simple formula in terms of \( n \) and \( p \) for \( \Pr \{ \text{not } P(x, y) \} \), for distinct vertices \( x \) and \( y \) in \( V \).

*Hint:* Use part (c), item 2.

(e) (2 points) What is the probability that two distinct vertices \( x \) and \( y \) lie on a three-cycle in \( G \)? Answer with a simple expression in terms of \( p \) and \( r \), where \( r := \Pr \{ \text{not } P(x, y) \} \) is the correct answer to part (d).

*Hint:* Express \( x \) and \( y \) being on a three-cycle as a simple formula involving \( E(x, y) \) and \( P(x, y) \).
Problem 13 (8 points). Sampling Concepts

Yesterday, the programmers at a local company wrote a large program. To estimate the fraction, \( b \), of lines of code in this program that are buggy, the QA team will take a small sample of lines chosen randomly and independently (so it is possible, though unlikely, that the same line of code might be chosen more than once). For each line chosen, they can run tests that determine whether that line of code is buggy, after which they will use the fraction of buggy lines in their sample as their estimate of the fraction \( b \).

The company statistician can use estimates of a binomial distribution to calculate a value, \( s \), for a number of lines of code to sample which ensures that with 97% confidence, the fraction of buggy lines in the sample will be within 0.006 of the actual fraction, \( b \), of buggy lines in the program.

Mathematically, the program is an actual outcome that already happened. The sample is a random variable defined by the process for randomly choosing \( s \) lines from the program. The justification for the statistician’s confidence depends on some properties of the program and how the sample of \( s \) lines of code from the program are chosen. These properties are described in some of the statements below. Indicate which of these statements are true:

- The probability that the ninth line of code in the program is buggy is \( b \).
- The probability that the ninth line of code in the sample is defective is \( b \).
- All lines of code in the program are equally likely to be the third line chosen in the sample.
- Given that the first line chosen in the sample is buggy, the probability that the second line chosen will also be buggy is greater than \( b \).
- Given that the last line in the program is buggy, the probability that the next-to-last line in the program will also be buggy is greater than \( b \).
- The expectation of the indicator variable for the last line in the sample being buggy is \( b \).
- Given that the first two lines of code selected in the sample are the same kind of statement (such as an assignment statement, a conditional statement, or a loop statement), the probability that the first line is buggy may be greater than \( b \).
- There is zero probability that all the lines in the sample will be different.
Problem 14 (9 points). Linear Recurrence & Expectation

Robbie the robot jumps around randomly on the integer line according to a peculiar set of rules. Letting $X_{n-1}$ be his position after $n-1$ flips, he again flips an unbiased coin to determine his next position, $X_n$. If the coin comes up heads, then $X_n = 3X_{n-1} - 2n$, and otherwise $X_n = X_{n-1} + 2$. Let $e_n := E[X_n]$ be Robbie’s expected position after $n$ steps.

(a) (2 points) Derive the equation

$$e_n = 2e_{n-1} - (n - 1),$$

for $n > 0$, explicitly indicating where your derivation uses such properties as linearity of expectation and/or the Law of Total Expectation.

(b) (2 points) Show that

$$\frac{2x - 1}{(1 - x)^2} = \sum_{n=0}^{\infty} (n - 1)x^n.$$
(c) (2 points) Let $E(x) = e_0 + e_1 x + e_2 x^2 + \cdots$ be the generating function for Robbie’s expected position after $n$ steps. Showing your derivation clearly, find the constant $k$ such that

$$(1 - 2x)E(x) - \frac{1 - 2x}{(1 - x)^2} = e_0 + k.$$ 

(d) (3 points) Circle the statement that best describes Robbie’s expected behavior. Show your work to receive partial credit.

- $|e_n| = \Theta(n)$
- $|e_n| = \Theta(n)$ if $X_0 < k$ and $|e_n| = \Theta(n^2)$ otherwise
- $|e_n| = \Theta(n^2)$
- $|e_n| = \Theta(2^n)$ if $X_0 \neq -k$ and $|e_n| = \Theta(n)$ otherwise
- $|e_n| = \Theta(2^n)$
- $|e_n| = \Theta(2^n)$ if $X_0 < k$ and $|e_n| = \Theta(n)$ otherwise
- $|e_n| = \Theta(2^n)$ if $X_0 \neq -k$ and $|e_n| = \Theta(n^2)$ otherwise
Appendix

Grid Networks

A grid is a simple communication network with low congestion. A grid with $n = 4$ inputs and outputs is pictured in the following figure.

![Grid Network Diagram]

**Theorem.** The congestion of an $n$-input array is at most 2.

**Proof.** Let $\pi$ be any permutation, and suppose $\pi(i) = j$. To route a packet from input $i$ to output $j$, use the path from input $i$ to the right along row $i$ until column $j$. Then continue the path downward along column $j$ to output $j$.

With this routing for $\pi$, the switch in row $k$ and column $l$ transmits at most two packets: the packet originating at input $k$ and the packet destined for column $l$.

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**The Pulverizer: example**

Euclid’s algorithm for finding the GCD of two numbers relies on repeated application of the equation:

$$\gcd(a, b) = \gcd(b, \text{rem}(a, b))$$

For example, we can compute the GCD of 259 and 70 as follows:

$$\begin{align*}
\gcd(259, 70) &= \gcd(70, 49) & \text{since } \text{rem}(259, 70) = 49 \\
&= \gcd(49, 21) & \text{since } \text{rem}(70, 49) = 21 \\
&= \gcd(21, 7) & \text{since } \text{rem}(49, 21) = 7 \\
&= \gcd(7, 0) & \text{since } \text{rem}(21, 7) = 0 \\
&= 7.
\end{align*}$$
The Pulverizer goes through the same steps, but requires some extra bookkeeping along the way: as we compute $\gcd(a, b)$, we keep track of how to write each of the remainders (49, 21, and 7, in the example) as a linear combination of $a$ and $b$ (this is worthwhile, because our objective is to write the last nonzero remainder, which is the GCD, as such a linear combination). For our example, here is this extra bookkeeping:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
<th>$\text{rem}(x,y) = x - q \cdot y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>259</td>
<td>70</td>
<td>49 = $259 - 3 \cdot 70$</td>
</tr>
<tr>
<td>70</td>
<td>49</td>
<td>21 = $70 - 1 \cdot 49$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>= $70 - 1 \cdot (259 - 3 \cdot 70)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>= $-1 \cdot 259 + 4 \cdot 70$</td>
</tr>
<tr>
<td>49</td>
<td>21</td>
<td>7 = $49 - 2 \cdot 21$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>= $(259 - 3 \cdot 70) - 2 \cdot (-1 \cdot 259 + 4 \cdot 70)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>= $3 \cdot 259 - 11 \cdot 70$</td>
</tr>
<tr>
<td>21</td>
<td>7</td>
<td>0</td>
</tr>
</tbody>
</table>

**RSA Public Key Encryption**

**Beforehand** The receiver creates a public key and a secret key as follows.

1. Generate two distinct primes, $p$ and $q$.
2. Let $n = pq$.
3. Select an integer $e$ such that $\gcd(e, (p - 1)(q - 1)) = 1$.
   The **public key** is the pair $(e, n)$. This should be distributed widely.
4. Compute $d$ such that $de \equiv 1 \pmod{(p - 1)(q - 1)}$.
   The **secret key** is the pair $(d, n)$. This should be kept hidden!

**Encoding** The sender encrypts message $m$ to produce $m'$ using the public key:

$$m' = \text{rem}(m^e, n).$$

**Decoding** The receiver decrypts message $m'$ back to message $m$ using the secret key:

$$m = \text{rem}((m')^d, n).$$
Generating Functions

The generating function for the infinite sequence \( \langle g_0, g_1, g_2, g_3, \ldots \rangle \) is the power series:

\[
G(x) = g_0 + g_1 x + g_2 x^2 + g_3 x^3 + \cdots .
\]

For example,

\[
\langle 0, 1, 2, 3, \ldots \rangle \leftrightarrow \frac{x}{(1 - x)^2} = \sum_{n=0}^{\infty} n x^n
\]

\[
\langle 1, 0, 1, 0, \ldots \rangle \leftrightarrow \frac{1}{1 - x^2} = \sum_{n=0}^{\infty} x^{2n}
\]

A “Useful” Generating Function

For any positive integer \( k \) and complex number \( \alpha \),

\[
\frac{1}{(1 - \alpha x)^k} = \sum_{n=0}^{\infty} \binom{n + k - 1}{k - 1} \alpha^n x^n.
\]

Convolution Counting

Let

\[
A(x) = \sum_{n=0}^{\infty} a_n x^n, \quad B(x) = \sum_{n=0}^{\infty} b_n x^n, \quad C(x) = A(x) \cdot B(x) = \sum_{n=0}^{\infty} c_n x^n.
\]

Then

\[
c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0.
\]

Rule (Convolution Rule). Let \( A(x) \) be the generating function for selecting items from set \( A \), and let \( B(x) \) be the generating function for selecting items from set \( B \). If \( A \) and \( B \) are disjoint, then the generating function for selecting items from the union \( A \cup B \) is the product \( A(x) \cdot B(x) \).

Finding a Generating Function for Fibonacci Numbers

The Fibonacci numbers are defined by:

\[
f_0 ::= 0
\]

\[
f_1 ::= 1
\]

\[
f_n ::= f_{n-1} + f_{n-2} \quad \text{for } n \geq 2
\]

Let \( F \) be the generating function for the Fibonacci numbers, that is,

\[
F(x) ::= f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + \cdots
\]
So we need to derive a generating function whose series has coefficients:

\[ (0, 1, f_1 + f_0, f_2 + f_1, f_3 + f_2, \ldots) \]

Now we observe that

\[
\begin{align*}
(0, 1, 0, 0, 0, \ldots) x \\
+ (0, f_0, f_1, f_2, f_3, \ldots) x F(x) \\
+ (0, 0, f_0, f_1, f_2, \ldots) x^2 F(x)
\end{align*}
\]

This sequence is almost identical to the right sides of the Fibonacci equations. The one blemish is that the second term is \(1 + f_0\) instead of simply 1. But since \(f_0 = 0\), the second term is ok.

So we have

\[
F(x) = x + x F(x) + x^2 F(x).
\]

\[
F(x) = \frac{x}{1 - x - x^2}.
\]  (1)

**Finding a Closed Form for the Coefficients**

Now we expand the righthand side of (1) into partial fractions. To do this, we first factor the denominator

\[
1 - x - x^2 = (1 - \alpha_1 x)(1 - \alpha_2 x)
\]

where \(\alpha_1 = \frac{1}{2}(1 + \sqrt{5})\) and \(\alpha_2 = \frac{1}{2}(1 - \sqrt{5})\) by the quadratic formula. Next, we find \(A_1\) and \(A_2\) which satisfy:

\[
F(x) = \frac{x}{1 - x - x^2} = \frac{A_1}{1 - \alpha_1 x} + \frac{A_2}{1 - \alpha_2 x}
\]  (2)

Now the coefficient of \(x^n\) in \(F(x)\) will be \(A_1\) times the coefficient of \(x^n\) in \(1/(1 - \alpha_1 x)\) plus \(A_2\) times the coefficient of \(x^n\) in \(1/(1 - \alpha_2 x)\). The coefficients of these fractions will simply be the terms \(\alpha_1^n\) and \(\alpha_2^n\) because

\[
\frac{1}{1 - \alpha_1 x} = 1 + \alpha_1 x + \alpha_1^2 x^2 + \cdots
\]

\[
\frac{1}{1 - \alpha_2 x} = 1 + \alpha_2 x + \alpha_2^2 x^2 + \cdots
\]

by the formula for geometric series.

So we just need to find \(A_1\) and \(A_2\). We do this by plugging values of \(x\) into (2) to generate linear equations in \(A_1\) and \(A_2\). It helps to note that from (2), we have

\[
x = A_1(1 - \alpha_2 x) + A_2(1 - \alpha_1 x),
\]

so simple values to use are \(x = 0\) and \(x = 1/\alpha_2\). We can then find \(A_1\) and \(A_2\) by solving the linear equations. This gives:

\[
A_1 = \frac{1}{\alpha_1 - \alpha_2} = \frac{1}{\sqrt{5}}
\]

\[
A_2 = -A_1 = -\frac{1}{\sqrt{5}}
\]
Substituting into (2) gives the partial fractions expansion of $F(x)$:

$$F(x) = \frac{1}{\sqrt{5}} \left( \frac{1}{1 - \alpha_1 x} - \frac{1}{1 - \alpha_2 x} \right).$$

So we conclude that the coefficient, $f_n$, of $x^n$ in the series for $F(x)$ is

$$f_n = \frac{\alpha_1^n - \alpha_2^n}{\sqrt{5}}$$

$$= \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right).$$