Problem Set 6 Solutions

Due: Monday, March 28 at 9 PM in Room 32-044

Problem 1. Sammy the Shark is a financial service provider who offers loans on the following terms.

- Sammy loans a client $m$ dollars in the morning. This puts the client $m$ dollars in debt to Sammy.
- Each evening, Sammy first charges a “service fee”, which increases the client’s debt by $f$ dollars, and then Sammy charges interest, which multiplies the debt by a factor of $p$. For example, if Sammy’s interest rate were a modest 5% per day, then $p$ would be 1.05.

(a) What is the client’s debt at the end of the first day?

Solution. At the end of the first day, the client owes Sammy $(m + f)p = mp + fp$ dollars.

(b) What is the client’s debt at the end of the second day?

Solution. $((m + f)p + f)p = mp^2 + fp^2 + fp$

(c) Write a formula for the client’s debt after $d$ days and find an equivalent closed form.

Solution. The client’s debt after three days is

$$(((m + f)p + f)p + f)p = mp^3 + fp^3 + fp^2 + fp.$$ 

Generalizing from this pattern, the client owes

$$mp^d + \sum_{k=1}^{d} fp^k$$

dollars after $d$ days. Applying the formula for a geometric sum gives:

$$mp^d + f \cdot \left( \frac{p^{d+1} - 1}{p - 1} - 1 \right)$$

Problem 2. Find closed-form expressions equal to the following sums. Show your work.
(a) \[ \sum_{i=0}^{n} \frac{9^i - 7^i}{11^i} \]

**Solution.** Split the expression into two geometric series and then apply the formula for the sum of a geometric series.

\[
\begin{align*}
\sum_{i=0}^{n} \frac{9^i - 7^i}{11^i} &= \sum_{i=0}^{n} \left( \frac{9}{11} \right)^i - \sum_{i=0}^{n} \left( \frac{7}{11} \right)^i \\
&= \frac{1 - \left( \frac{9}{11} \right)^{n+1}}{1 - \frac{9}{11}} - \frac{1 - \left( \frac{7}{11} \right)^{n+1}}{1 - \frac{7}{11}} \\
&= -\frac{11}{2} \cdot \left( \frac{9}{11} \right)^{n+1} + \frac{11}{4} \cdot \left( \frac{7}{11} \right)^{n+1} + \frac{11}{4}
\end{align*}
\]

(b) \[ \prod_{i=1}^{n} 3^{4i+5} \]

**Solution.** Taking the logarithm reduces this product to an easy sum.

\[
\prod_{i=1}^{n} 3^{4i+5} = 3^{\log_3 \left( \prod_{i=1}^{n} 3^{4i+5} \right)} = 3^{\sum_{i=1}^{n} 4i+5} = 3^{2n(n+1)+5n}
\]

(c) \[ \sum_{j=1}^{n} \sum_{i=0}^{\infty} j^{5/3} \cdot \left( 1 - \frac{1}{2j^{1/3}} \right)^i \]

**Solution.** This fearsome-looking sum is a paper tiger; we just apply the formula for the sum of a geometric series followed by the formula for the sum of an arithmetic series.

\[
\begin{align*}
\sum_{j=1}^{n} \sum_{i=0}^{\infty} j^{5/3} \cdot \left( 1 - \frac{1}{2j^{1/3}} \right)^i &= \sum_{j=1}^{n} j^{5/3} \cdot \frac{1}{1 - \left( 1 - \frac{1}{2j^{1/3}} \right)} \\
&= \sum_{j=1}^{n} 2j^2 \\
&= \frac{2n(n + \frac{1}{2})(n + 1)}{3}
\end{align*}
\]
Problem 3. There is a bug on the edge of a 1-meter rug. The bug wants to cross to the other side of the rug. It crawls at 1 cm per second. However, at the end of each second, a malicious first-grader named Mildred Anderson stretches the rug by 1 meter. Assume that her action is instantaneous and the rug stretches uniformly. Thus, here’s what happens in the first few seconds:

- The bug walks 1 cm in the first second, so 99 cm remain ahead.
- Mildred stretches the rug by 1 meter, which doubles its length. So now there are 2 cm behind the bug and 198 cm ahead.
- The bug walks another 1 cm in the next second, leaving 3 cm behind and 197 cm ahead.
- Then Mildred strikes, stretching the rug from 2 meters to 3 meters. So there are now $3 \cdot (3/2) = 4.5$ cm behind the bug and $197 \cdot (3/2) = 295.5$ cm ahead.
- The bug walks another 1 cm in the third second, and so on.

Your job is to determine this poor bug’s fate.

(a) During second $i$, what fraction of the rug does the bug cross?

**Solution.** During second $i$, the length of the rug is $100i$ cm and the bug crosses 1 cm. Therefore, the fraction that the bug crosses is $1/100i$.

(b) Over the first $n$ seconds, what fraction of the rug does the bug cross altogether?

**Solution.** The bug crosses $1/100$ of the rug in the first second, $1/200$ in the second, $1/300$ in the third, and so forth. Thus, over the first $n$ seconds, the fraction crossed by the bug is:

$$\sum_{k=1}^{n} \frac{1}{100k} = H_n/100$$

(This formula is valid only until the bug reaches the far side of the rug.)

(c) Approximately how many seconds does the bug need to cross the entire rug?

**Solution.** The bug arrives at the far side when the fraction it has crossed reaches 1. This occurs when $n$, the number of seconds elapsed, is sufficiently large that $H_n/100 \geq 1$. Now $H_n$ is approximately $\ln n$, so the bug arrives about when:

$$\frac{\ln n}{100} \geq 1$$

$$\ln n \geq 100$$

$$n \geq e^{100} \approx 10^{43} \text{ seconds}$$
Problem 4. Use integration to find lower and upper bounds on the following infinite sum that differ by at most $0.1$. Show your work.

$$ S = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots $$

To achieve this accuracy, add up the first few terms explicitly and then use integration to bound all remaining terms.

Solution. The sum of the first three terms is:

$$ S = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} = \frac{49}{36} $$

An upper bound on the remaining terms is:

$$ \int_{3}^{\infty} \frac{1}{x^2} \, dx = \frac{1}{3} $$

And a lower bound is:

$$ \int_{3}^{\infty} \frac{1}{(x+1)^2} \, dx = \frac{1}{4} $$

Overall, we have:

$$ \frac{58}{36} \leq \frac{49}{36} + \frac{1}{4} \leq S \leq \frac{49}{36} + \frac{1}{3} = \frac{61}{36} $$

These bounds differ by $1/12 < 0.1$. The actual value of the sum is $\pi^2/6$, though the proof is not easy.

Problem 5. A seasoned MIT undergraduate can:

- Complete a problem set in 2 days.
- Write a paper in 2 days.
- Take a 2-day road trip.
- Study for an exam in 1 day.
- Play foosball for an entire day.

An $n$-day schedule is a sequence of activities that require a total of $n$ days. For example, here are three possible 7-day schedules:

pset, paper, pset, foosball
paper, study, foosball, pset, study
road trip, road trip, road trip, study
(a) Express the number of possible \( n \)-day schedules using a recurrence equation and sufficient base cases.

**Solution.**

\[
S(0) = 1, \\
S(1) = 2.
\]

Any schedule for \( n > 1 \) days ends with one of 3 possible 2-day activities or one of 2 possible 1-day activities. So

\[
S(n) = 2S(n-1) + 3S(n-2) \quad \text{for } n > 1.
\]

(b) Find a closed-form expression for the number of possible \( n \)-day schedules by solving the recurrence.

**Solution.** The characteristic polynomial for this linear homogeneous recurrence is \( x^2 - 2x - 3 = (x + 1)(x - 3) \). Hence the solution is of the form \( S(n) = a(-1)^n + b3^n \). Letting \( n = 0 \), we conclude that \( a + b = 1 \), and letting \( n = 1 \), we conclude \( -a + 3b = 2 \), so \( b = 3/4, a = 1/4 \), and the solution is:

\[
S(n) = \frac{3^{n+1} + (-1)^n}{4}.
\]

**Problem 6.** Find a closed-form expression for \( T(n) \), which is defined by the following recurrence:

\[
T(0) = 0 \\
T(1) = 1 \\
T(n) = 5T(n-1) - 6T(n-2) + 6 \quad \text{for all } n \geq 2
\]

**Solution.** The characteristic equation is \( x^2 - 5x + 6 = 0 \), which has roots \( x = 2 \) and \( x = 3 \). Thus, the homogenous solution is:

\[
T(n) = A \cdot 2^n + B \cdot 3^n
\]

For a particular solution, let’s first guess \( T(n) = c \):

\[
c = 5c - 6c + 6 \\
\Rightarrow c = 3
\]

Our guess was correct; \( T(n) = 3 \) is a particular solution. Adding this to the homogenous solution gives the general solution:

\[
T(n) = A \cdot 2^n + B \cdot 3^n + 3
\]
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Substituting \( n = 0 \) and \( n = 1 \) gives:

\[
\begin{align*}
0 &= A + B + 3 \\
1 &= 2A + 3B + 3
\end{align*}
\]

Solving this system gives \( A = -7 \) and \( B = 4 \). Therefore:

\[
T(n) = -7 \cdot 2^n + 4 \cdot 3^n + 3
\]

**Problem 7.** Determine which of these choices

\[
\Theta(n), \quad \Theta(n^2 \log n), \quad \Theta(n^2), \quad \Theta(1), \quad \Theta(2^n), \quad \Theta(2^{n \ln n}), \quad \text{none of these}
\]

describes each function’s asymptotic behavior. Proofs are not required, but briefly explain your answers.

(a) \( n + \ln n + (\ln n)^2 \)

**Solution.** Both \( n > \ln n \) and \( n > (\ln n)^2 \) hold for all sufficiently large \( n \). Thus, for all sufficiently large \( n \):

\[
n < n + \ln n + (\ln n)^2 < n + n + n
\]

So \( n + \ln n + (\ln n)^2 = \Theta(n) \).

(b) \( \frac{n^2 + 2n - 3}{n^2 - 7} \)

**Solution.** Observe that:

\[
\lim_{n \to \infty} \frac{n^2 + 2n - 3}{n^2 - 7} = 1
\]

This means, that for all sufficiently large \( n \), the fraction lies, for example, between, 0.99 and 1.01 and is therefore \( \Theta(1) \).

(c) \( \sum_{i=0}^{n} 2^{2i+1} \)

**Solution.** Geometric sums are dominated by their largest term, which is \( 2^{2n+1} = 2 \cdot 4^n \). This is \( \Theta(4^n) \), which does not appear in the list provided.

(d) \( \ln(n^2!) \)

**Solution.** By Stirling’s formula:

\[
n^2! \sim \sqrt{2\pi n^2} \left( \frac{n^2}{e} \right)^{n^2}
\]
Taking logarithms gives:

\[
\ln(n^2!) \sim \ln(\sqrt{2\pi n^2} \left(\frac{n^2}{e}\right)^{n^2})
\]

\[= \ln(\sqrt{2\pi n^2}) + \ln \left(\frac{n^2}{e}\right)^{n^2}\]

The first term is tiny compared to the second, which we can rewrite as:

\[
\ln \left(\frac{n^2}{e}\right)^{n^2} = n^2 \ln \left(\frac{n^2}{e}\right) = \Theta(n^2 \ln n)
\]

(e)

\[
\sum_{k=1}^{n} k \left(1 - \frac{1}{2k}\right)
\]

Solution. The expression in parentheses is always at least 1/2 and at most 1. Thus, we have the bounds:

\[
\frac{1}{2} \sum_{k=1}^{n} k \leq \sum_{k=1}^{n} k \left(1 - \frac{1}{2k}\right) \leq \sum_{k=1}^{n} k
\]

Since the first expression and the last are both \(\Theta(n^2)\), so is the one in the middle.

**Problem 8.** A triangular number is an integer \(n\) of the form

\[n = 1 + 2 + 3 + \ldots + k = \frac{k(k + 1)}{2}\]

where \(k\) is a positive integer.

(a) Describe a solution to the four-peg Towers of Hanoi puzzle with \(k(k + 1)/2\) disks that requires \(T_k\) moves, where:

\[
T_1 = 1
\]

\[
T_k = 2T_{k-1} + 2^k - 1
\]

Solution.

- Move all but the \(k\) largest disks to another peg recursively. This requires \(T(k - 1)\) moves.
- Move the \(k\) largest disks to another peg using the three-peg strategy. This requires \(2^k - 1\) moves.
- Now move all the other disks on top of the \(k\) largest disks recursively. This requires \(T(k - 1)\) moves.
Thus, with this strategy, the total number of moves required to move a stack of \( k(k+1)/2 \) disks is \( T(k) = 2T(k-1) + 2^k - 1 \).

(b) Find a closed form expression equal to \( T_k \).

Solution. This is an inhomogenous linear equation. Let’s begin by trying to find a particular solution. There is both an exponential term \( (2^k) \) and a constant term, so we might guess something of the form \( a2^k + c \):

\[
a2^k + c = 2(a2^{k-1} + c) + 2^k - 1
\]

\[
= (a + 1)2^k + 2c - 1
\]

\[
\Rightarrow 0 = 2^k + (c - 1)
\]

Evidently, the constant term is \( c = 1 \), but the exponential part is more complicated. Our recipe says we should next try a particular solution of the form \( a2^k + bk2^k + 1 \):

\[
a2^k + bk2^k + 1 = 2(a2^{k-1} + b(k-1)2^{k-1} + 1) + 2^k - 1
\]

\[
= (a - b + 1)2^k + bk2^k - 1
\]

Equating the coefficients of the \( 2^k \) terms gives \( a = a - b + 1 \), which implies \( b = 1 \). Thus, \( a2^k + k2^k + 1 \) is a particular solution for all \( a \). As long as we have this degree of freedom, we might as well choose \( a \) so this solution is consistent with the boundary condition \( T_1 = 1 \) and be done:

\[
a2^1 + 1 \cdot 2^1 + 1 = 1 \quad \Rightarrow \quad a = -1
\]

Therefore, the solution to the recurrence is \( T_k = (k - 1)2^k + 1 \).

(c) Approximately how many moves are required to solve the four-peg, \( n \)-disk Towers of Hanoi puzzle as a function of \( n \)? Assume \( n \) is a triangular number. (For style points, make correct use of asymptotic notation.)

Solution. We have \( k = \frac{1}{2}(\sqrt{8n + 1} - 1) = \sqrt{2n} + O(1) \). So the number of moves required is \( \Theta(\sqrt{n2\sqrt{2n}}) \).