**Problem 1.** Let $p$ be the string $()$. A string of parentheses is said to be *erasable* iff it can be reduced to the empty string by repeatedly erasing occurrences of $p$. For example, here's how to erase the string $((())())()$:

$$((())())() \to (()) \to () \to \lambda.$$

On the other hand the string $()) (((((())$ is not erasable because when we try to erase, we get stuck at $)(((:$

$$())((((()) \to )(((() \to )((( \not\to$$

Let $\text{Erasable}$ be the set of erasable strings of parentheses. Let $\text{RecMatch}$ be the recursive data type of strings of *matched* parentheses. (The definition of $\text{RecMatch}$ is repeated in the Appendix.)

**(a)** Use structural induction to prove that

$$\text{RecMatch} \subseteq \text{Erasable}\,.$$

**(b)** Supply the missing parts of the following proof that

$$\text{Erasable} \subseteq \text{RecMatch}\,.$$

*Proof.* We prove by induction on the length, $n$, of strings, $x$, that if $x \in \text{Erasable}$, then $x \in \text{RecMatch}$. The induction predicate is

$$P(n) ::= \forall x \in \text{Erasable}\,.\,[|x| \leq n \ \text{ IMPLIES } \ x \in \text{RecMatch}]$$

**Base case**:

**What is the base case? Prove that $P$ is true in this case.**

**Inductive step**: To prove $P(n+1)$, suppose $|x| \leq n+1$ and $x \in \text{Erasable}$. We need only show that $x \in \text{RecMatch}$. Now if $|x| < n+1$, then the induction hypothesis, $P(n)$, implies that $x\,\text{RecMatch}$, so we only have to deal with $x$ of length exactly $n+1$.

Let's say that a string $y$ is an *erase* of a string $z$ iff $y$ is the result of erasing a single occurrence of $p$ in $z$.

Since $x \in \text{Erasable}$ and has positive length, there must be an erase, $y \in \text{Erasable}$, of $x$. So $|y| = n-1$, and since $y \in \text{Erasable}$, we may assume by induction hypothesis that $y \in \text{RecMatch}$.

Now we argue by cases:

**Case** [$y$ is the empty string].

**Prove that $x \in \text{RecMatch}$ in this case.**

**Case** [$y = (s)t$ for some strings $s, t \in \text{RecMatch}$.] Now we argue by subcases.

- **Subcase** [$x$ is of the form $(s')t$ where $s$ is an erase of $s'$].
  Since $s \in \text{RecMatch}$, it is erasable by part (b), which implies that $s' \in \text{Erasable}$. But $|s'| < |x|$, so by induction hypothesis, we may assume that $s' \in \text{RecMatch}$. This shows that $x$ is the result of the constructor step of $\text{RecMatch}$, and therefore $x \in \text{RecMatch}$.
- **Subcase** [$x$ is of the form $(s)t'$ where $t$ is an erase of $t'$].
  **Prove that $x \in \text{RecMatch}$ in this subcase.**

- **Subcase**[$x = p(s)t$].
  **Prove that** $x \in \mathrm{RecMatch}$ **in this subcase.**

The proofs of the remaining subcases are just like this last one. **List these remaining subcases.**

This completes the proof by induction on $n$, so we conclude that $P(n)$ holds for all $n \in \mathbb{N}$. Therefore $x \in \mathrm{RecMatch}$ for every string $x \in \mathrm{Erasable}$. That is,

$$\mathrm{Erasable} \subseteq \mathrm{RecMatch} \text{ and hence } \mathrm{Erasable} = \mathrm{RecMatch}\,.$$

$\square$