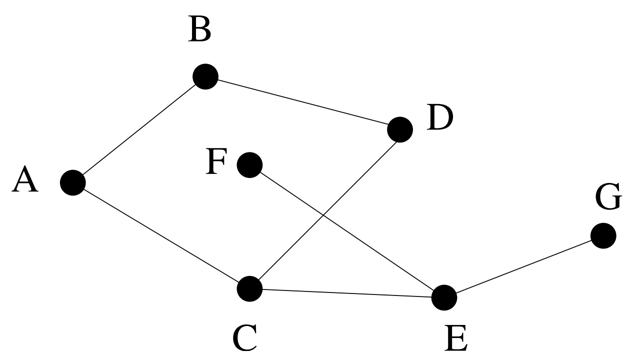
Notes for Recitation 6

1 Graph Basics

Let $G = (V, E)$ be a graph. Here is a picture of a graph.



Recall that the elements of V are called vertices, and those of E are called edges. In this example the vertices are $\{A, B, C, D, E, F, G\}$ and the edges are

$$\{A—B, B—D, C—D, A—C, E—F, C—E, E—G\}.$$

Definition 1. *Let $G = (V, E)$ be a graph. A* path *in G is a sequence of vertices*

$$v_0, \ldots, v_k$$

with $k \geq 0$ such that v_i—v_{i+1} is an edge in E for all $i \geq 0$ such that $i < k$, and all the v_i's, except possibly v_0 and v_k, are different. That is, if $0 \leq i < j \leq k$, then $v_i = v_j$ only if both $i = 0$ and $j = k$. The path is said to start *at v_0, to* end *at v_k, and* length *of the path is defined to be k. If it happens that $v_0 = v_k$, then we say the path is* closed. *In this case we call the path a* cycle.

For example, the graph in the figure above has a length 5 path A,B,D,C,E,G. There is a cycle of length 4 – namely A,B,D,C,A.

The *distance* between two vertices is just the length of the shortest path between them. So in the figure above, the distance $d(A, D)$ between A and D is 2. The *diameter* of a graph is the largest distance between any two vertices. In the figure above, the diameter is 4 since the distance between B and G is 4, and no other pair has larger distance (B and F also have distance 4).

Definition 2. *A* walk *in a graph G is an alternating sequence of vertices and edges of the form:*

$$v_0, v_0—v_1, v_1, v_1—v_2, v_2, \ldots, v_{n-1}, v_{n-1}—v_n, v_n$$

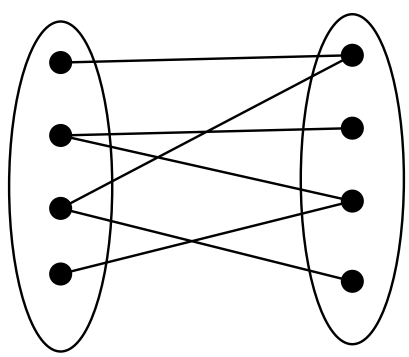
If $v_0 = v_n$, then the walk is closed. In this case we say the walk is a circuit.

Walks are similar to paths. However, a walk can cross itself, traverse the same edge multiple times, etc. There is a walk between two vertices if and only if there is a path between the vertices. The length of a walk is defined in the same way as the length for a path. It is just the number of edges on the walk. So, in the above graph A, C, D, C is a walk, but not a path. Moreover, A, C, D, C, A is a circuit.

2 Bipartite Graphs

Now we're going to talk about *coloring* a graph, which means assigning a color to each of its vertices. We will look at the very special case when we only have two colors to work with, say, black and white. A graph G is *bipartite* if each vertex can be assigned one of the two colors so that the endpoints of each edge in the graph are colored differently.

Then we can put all the black vertices in a clump on the left and all the white vertices in a clump on the right. Since every edge joins differently-colored vertices, every edge must run between the two clumps. Therefore, every bipartite graph looks something like this:



Bipartite graphs are both useful and common. In this problem, you will prove:

Theorem. *A graph G is 2-colorable iff it contains no odd length cycle.*

As usual with "iff" assertions, the proof splits into two proofs: part (a) asks you to prove that the left side of the "iff" implies the right side. The other problem parts prove that the right side implies the left.

1. Assume the left side and prove the right side. Three to five sentences should suffice.

   Solution. First, we assume that G is 2-colorable and prove that G contains no odd length cycle.

   Select a 2-coloring of G. Consider an arbitrary cycle with successive vertices $v_1, v_2, \ldots, v_k, v_1$. Then the vertices v_i must be one color for all even i and the other color for all odd i. (We could confirm this claim with a proof by induction, but it seems obvious enough to accept without further proof.) Since v_1 and v_k must be colored differently, k must be even. Thus, the cycle has even length. We can make the same argument for any cycle in G, so every cycle has even length.

2. Now assume the right side. As a first step toward proving the left side, explain why we can focus on a single connected component H within G.

   Solution. We assume that G contains no odd cycle and prove that G is 2-colorable. If we can 2-color every connected component of G, then we can 2-color all of G since vertices in different connected components have no edges in between them. Thus, it suffices to show that an arbitrary connected component H of G is 2-colorable.

3. Thus, we may assume that G is connected for the remainder of the proof. Let v be an arbitrary vertex in G. Explain why if G is not bipartite, then we can find two vertices a and b and an edge (a, b) in G, such that $d(v, a) \equiv d(v, b) \bmod 2$.

   Solution. We prove this claim by contradiction. Suppose this were not true, and suppose we color all vertices at an even distance from v one color, and all vertices at an odd distance from v another color. If we can't find an edge (a, b) where $d(v, a) \equiv d(v, b) \bmod 2$, then G would be bipartite.

4. So assume we've found a and b as above. Our goal will be to show that G has an odd cycle. Let $P(v, a)$ and $P(v, b)$ be *shortest* paths from v to a and from v to b, respectively. Explain why the walk $P(v, a)$ reverse$(P(v, b))$ is a circuit of odd length, where reverse$(v_1, \ldots, v_k)$ is defined to be $v_k, \ldots, v_1$.

   Solution. The walk starts and ends at the vertex v, so by definition it is a circuit. Its length is the length of $P(v, a)$ plus that of $P(v, b)$, plus 1 to account for the edge (a, b). Since $P(v, a)$ and $P(v, b)$ are shortest paths, their lengths are $d(v, a)$ and $d(v, b)$, respectively. It follows that modulo 2, the length of the walk is $d(v, a) + d(v, b) + 1 \equiv 1 \bmod 2$, since $d(v, a) \equiv d(v, b) \bmod 2$.

5. Explain how to find an odd-length cycle from the circuit $P(v, a)$ reverse$(P(v, b))$. Hint: consider the vertices in common to both $P(v, a)$ and $P(v, b)$.

   Solution. Consider the vertices which occur in both $P(v, a)$ and $P(v, b)$, and let w be the one for which $d(v, w)$ is largest. Such a w must exist (note that w may equal v) and it is unique. Indeed, if w and w' were the same distance from v, they could not both be on $P(v, a)$, as otherwise $P(v, a)$ would not be a shortest path.

   Let $P(w, a)$ be the path which starts at w, and follows $P(v, a)$ until reaching a. That is, $P(w, a)$ is the subpath of $P(v, a)$ from vertices w to a. Similarly define $P(w, b)$ as the subpath of $P(v, b)$ from vertices w to b.

The claim is that $P(w, a)$ reverse$(P(w, b))$ is a cycle of odd length. First, we show that it is in fact a cycle. It is clearly a circuit, so we just need to show there are no repeated vertices. But this follows from the way that $P(w, a)$ and $P(w, b)$ were constructed. Their only vertex in common is w, as otherwise w could not be the vertex common to $P(v, a)$ and $P(v, b)$ with largest distance from v. Next, we show that it has odd length. The length of $P(w, a)$ is $d(v, a) - d(v, w)$ since $P(w, a)$ is a shortest path from w to a (if this were not the case, then $P(v, a)$ could not have been a shortest path from v to a), and similarly the length of $P(w, b)$ is $d(v, b) - d(v, w)$. It follows that the length of $P(w, a)$ reverse$(P(w, b))$ is

$$d(v, a) - d(v, w) + 1 + d(v, b) - d(v, w) = d(v, a) + d(v, b) + 1 - 2d(v, w).$$

Thus, its length modulo 2 is $d(v, a) + d(v, b) + 1 \equiv 1 \bmod 2$ since $d(v, a) \equiv d(v, b) \bmod 2$. This completes the proof.

3 Euler Circuits

Now we'll consider a special kind of circuit called an *Euler circuit*, named after the famous mathematician Leonhard Euler. You may have encountered Euler earlier. Probably not on the street, but maybe through his famous constant $e \approx 2.718$.

Euler considered circuits of graphs of a very special form.

Definition 3. *An Euler circuit of a graph G is a circuit of G which visits every edge exactly once.*

Does the graph in the figure of Section 1 contain an Euler circuit? Well, if it did, the edge (E, F) would need to be included. If the walk does not start at F then at some point it traverses edge (E, F), and now it is stuck at F since F has no other edges incident to it and an Euler circuit can't traverse (E, F) twice. But then the walk could not be a circuit. On the other hand, if the walk starts at F, it must then go to E along (E, F), but now it cannot return to F. It again cannot be a circuit. This argument generalizes to show that if a graph has a vertex of degree 1, it cannot contain an Euler circuit.

On the other hand, it is easy to see that any cycle contains an Euler circuit. You can just start at any vertex and walk around back to it.

Naturally, this leads us to the question of which graphs contain Euler circuits. At first glance this may seem like a daunting problem. Nevertheless, you will now completely solve it. Recall that the *degree* of a vertex is the number of edges adjacent to it.

1. Show that if a graph G has an Euler circuit, then the degree of each of its vertices is even.

   Solution. Consider any Euler circuit $C = v_1, v_2, \ldots, v_r, v_1$ of G. Consider any vertex v. Then every time v occurs in C, there is a vertex a which comes immediately before

v and a vertex b which comes immediately after v. Note that this holds for $v = v_1$ as well since C is a circuit. Moreover, (a, v) and (v, b) must be distinct edges of G since C is an Euler circuit. It follows that if v occurs s times in C, then it has degree $2s$ since every edge incident to v occurs in C exactly once. Thus, v has even degree.

2. We will now show that if the degree of each of the vertices of a connected graph G is even, then G has an Euler circuit. To do this, consider a longest walk $W = v_0, v_1, \ldots, v_r$ in G using no edge more than once. Show that W is a circuit.

   Solution. Since W is a longest walk, all edges incident to v_r must already appear in W. It follows that $v_r = v_0$, as otherwise by the same argument given above, v_r would have odd degree.

3. Suppose that W is not an Euler circuit. Show that this implies it cannot be a longest walk in which each edge is visited at most once. Note that this yields a contradictionto our choice of W.

   Solution. Since G is connected, there is an edge $e = (v, v_i)$ not visited by W which is incident to some vertex v_i in W. Consider the walk

   $$W' = v, v_i, v_{i-1}, \ldots, v_0, v_{r-1}, v_{r-2}, \ldots, v_{i+1}.$$

   This walk uses each edge at most once and is of length longer than W.