Problems for Recitation 13 (5/8/98)

1 Class Exercise

Justify the lines in the proof of Theorem 1.2 with the choices given below. Select all those apply.

1. Definition of Expectation
2. Definition of Variance
3. Definition of Covariance
4. Linearity of Expectation
5. Product of Independent Random Variables

Theorem 1.2: For random variables R_1 and R_2,

$$\mathrm{Var}\left[R_1 + R_2\right] = \mathrm{Var}\left[R_1\right] + \mathrm{Var}\left[R_2\right] + 2\mathrm{Cov}\left[R_1, R_2\right].$$

Proof.

$$\begin{aligned}
\mathrm{Var}\left[R_1 + R_2\right] &= \mathrm{E}\left[(R_1 + R_2)^2\right] - \mathrm{E}^2\left[R_1 + R_2\right] \quad (\qquad) \\
&= \mathrm{E}\left[R_1^2 + 2R_1R_2 + R_2^2\right] - \left(\mathrm{E}\left[R_1\right] + \mathrm{E}\left[R_2\right]\right)^2 \quad (\qquad) \\
&= \mathrm{E}\left[R_1^2\right] + 2\mathrm{E}\left[R_1R_2\right] + \mathrm{E}\left[R_2^2\right] - \mathrm{E}^2\left[R_1\right] - 2\mathrm{E}\left[R_1\right]\mathrm{E}\left[R_2\right] - \mathrm{E}^2\left[R_2\right] \quad (\qquad) \\
&= \left(\mathrm{E}\left[R_1^2\right] - \mathrm{E}^2\left[R_1\right]\right) + \left(\mathrm{E}\left[R_2^2\right] - \mathrm{E}^2\left[R_2\right]\right) + 2(\mathrm{E}\left[R_1R_2\right] - \mathrm{E}\left[R_1\right]\mathrm{E}\left[R_2\right]) \quad (\qquad) \\
&= \mathrm{Var}\left[R_1\right] + \mathrm{Var}\left[R_2\right] + 2\mathrm{Cov}\left[R_1, R_2\right] \quad (\qquad)
\end{aligned}$$

$\square$

2 Team Problems

Problem 1. The Petersburg Casino offers the following game: the gambler bets a fixed wager, and then the dealer flips a fair coin (dealers do not flip coins in US casinos, but they do in Petersburg) until it comes up heads. The gambler receives \$1 if the coin shows heads the first time, \$2 if it shows the first head at second toss, and in general $\$2^{k-1}$ if the dealer tosses the coin k times to get the first head. The fixed wager is \$100,000.

(a) What is the expected amount of money that the Petersburg Casino is going to pay to a gambler that plays this game? What about the variance? Is it reasonable for the gambler to risk \$100,000 dollars in a game where there is a 50% chance of getting only *one dollar* back (this is not really a mathematical question: just think about that.)

(b) What is the probability that the gambler does not lose money in this game, i.e. what is the probability that he gets back from the Casino more than \$100,000?

Problem 2. A Harvard student, lost in the underground corridors of MIT, finds himself in a room containing three doors. Door 1 leads him to freedom (out of MIT) after one hour travel; door 2 returns him to the same room after four hours travel, and door 3 returns him to the room after seven hours travel. Each time he finds himself in the room, the Harvard student picks a door with equal probability. Let T denote the time until he finds the exit.

(a) Define a sequence of independent identically distributed random variables R_i and a stopping rule Q such that

$$T = \sum_{i=1}^{Q} R_i$$

(b) Is R_k independent of the event that $Q \geq k$? Is R_k independent of Q?

(c) If it is applicable, use Wald's Theorem to find $\mathrm{E}[T]$.

(d) Now suppose that an MIT student is also lost and finds herself in a similar situation to the Harvard student. The MIT student never chooses the same door twice. What is $\mathrm{E}[T]$ for the MIT student?

Problem 3. If a random variable has infinite expectation, then the variance is undefined, however a random variable may have finite expectation and infinite variance.

(a) Show an example of a random variable whose expectation is finite but whose variance is infinite.

[It might be easier to use a random variable whose expectation is zero.]

(b) If you used a random variable which can assume negative values in answering the previous question, try to find an example of a *non-negative* random variable whose expectation is finite and whose variance is infinite.

[Hint: notice that $\sum_{n=1}^{\infty} 1/2^{2n} = 1/3$ and $\sum_{n=1}^{\infty} 1/2^{n} = 1$.]

Problem 4. Let X be a non-negative random variable with expectation $n/2$. Markov's inequality implies that

$$\Pr\left[X \geq \frac{2}{3}n\right] \leq \frac{3}{4}. \tag{1}$$

(a) Define a random variable X with expectation $n/2$ such that the bound (1) holds with equality.

(b) If $X_1, \ldots, X_n$ are a collection of *independent* unbiased Bernoulli variables over the same sample space, then $X = \sum_i X_i$ has expectation $n/2$ and is non-negative, therefore bound 1 applies. Unfortunately it is exceedingly inaccurate. However, under the general hypotheses for Markov's inequality (where no independence is assumed) the bound (1) is the best one can hope for.

Define a sample space and unbiasied Bernoulli variables $X_1, \ldots, X_n$ on it, such that for $X = \sum_{i=1}^{n} X_i$ the bound (1) holds with equality (assume n is a multiple of 3).

[Hint: the variables will have to be dependent. The event $X_1 = \cdots = X_n = 0$ has probability 1/4 in the sample space.]