Problem for Recitation 9 (4/10/98)

Problem 1. Now we consider another ingenious approach to counting well-matched parenthesis strings. Our approach will involve considering *paths* in the plane from a point (x_1, y_1) to (x_2, y_2) for $x_i, y_i \in \mathbb{Z}$, where each step in the path is at a ± 45 degree angle, increasing the x-coordinate by 1 and the y-coordinate by ± 1. For example, a path might start at $(1, 1)$ and make seven steps to the successive points $(2, 0)$, $(3, -1)$, $(4, 0)$, $(5, 1)$, $(6, 2)$, $(7, 1)$, $(8, 0)$ as in Figure 1.

(a) Explain why the number of well-matched parenthesized strings of length $2n$ is the same as the number of paths from $(0, 0)$ to $(2n, 0)$ which only touch the x-axis at their endpoints. (That is, if $0 < i < 2n$ and (i, j) is a point on the path, then $j > 0$.)

(b) How many paths are there from (x_1, y_1) to (x_2, y_2)? (Express the answer as a binomial coefficient.)

(c) For $y_1, y_2 > 0$, show that the number of paths from (x_1, y_1) to (x_2, y_2) which touch or cross the x-axis is equal to the number of paths from $(x_1, -y_1)$ to (x_2, y_2). See Figure 2.

(d) Conclude that the number of well-matched strings of $2n$ parentheses equals the $n - 1$st Catalan number

$$\frac{1}{n}\binom{2n-2}{n-1}.$$

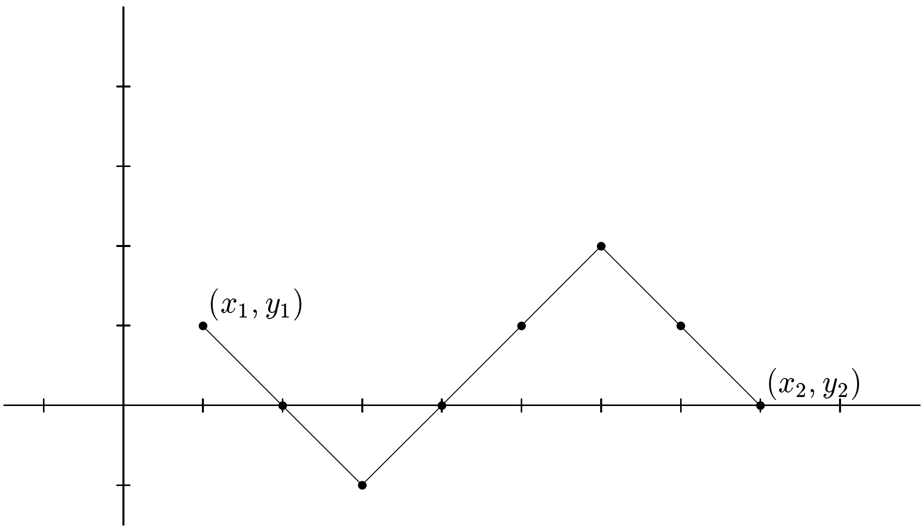

Figure 1: *A path from* $(1, 1)$ *to* $(8, 0)$.

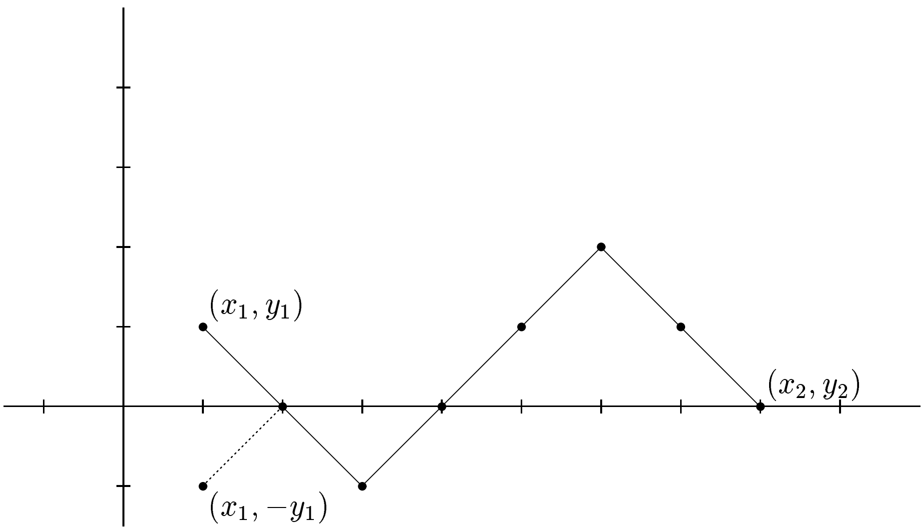

Figure 2: *Reflection of the initial part of a path.*