Massachusetts Institute of Technology
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Notes for Recitation 8 (4/3/98)

1 Binomial coefficients

In lecture, we saw the *binomial coefficients*

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

These represent the number of ways to choose r elements from a set of n elements, where the order does not matter. Here we see more properties of binomial coefficients.

Binomial coefficients are related to Pascal's triangle.[1]

Pascal's Triangle

$$\begin{array}{c} 1 \\ 1\ 1 \\ 1\ 2\ 1 \\ 1\ 3\ 3\ 1 \\ 1\ 4\ 6\ 4\ 1 \\ 1\ 5\ 10\ 10\ 5\ 1 \\ \cdots \end{array}$$

Define $P_{n,r}$ as the r-th entry in the n-th row, where $n \geq 0$ and $0 \leq r \leq n$ (so we count rows and entries starting from the 0-th.) Then, the inductive definition of $P_{n,r}$ is as follows: for any $n \geq 0$,

$$P_{n,r} = \begin{cases} 1 & \text{If } r = 0 \\ 1 & \text{If } r = n \\ P_{n-1,r} + P_{n-1,r-1} & \text{If } 1 \leq r \leq n-1 \ . \end{cases}$$

We now show that binomial coefficients satisfy a similar inductive property.

[1] A curiosity: Pascal's triangle is called "Tartaglia's triangle" in Italy.

Lemma 1.1 *For* $1 \leq r \leq n-1$,

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

Proof. [Combinatorial] $\binom{n}{r}$ is the number of subsets of size r chosen from a set of size n. Fix some element x in a set S of size n. There are $\binom{n-1}{r}$ subsets of S with r elements which contain x and there are $\binom{n-1}{r-1}$ subsets of S with r elements which do not contain x. Therefore, the total number of subsets of S with r elements is $\binom{n-1}{r-1} + \binom{n-1}{r-1}$. ∎

Proof. [Algebraic]

$$\begin{aligned}
\binom{n-1}{r} + \binom{n-1}{r-1} &= \frac{(n-1)!}{(n-r-1)!r!} + \frac{(n-1)!}{(n-r)!(r-1)!} \\
&= (n-r)(n-1)!(n-r-1)!r! + \frac{(n-1)!}{(n-r)!(r-1)!} \\
&= \frac{((n-r)+r)(n-1)!}{(n-r)!r!} \\
&= \frac{n!}{(n-r)!r!}\ .
\end{aligned}$$

∎

It is now easy to prove by induction that $P_{n,r}$ is equal to $\binom{n}{r}$.

Theorem 1.2 $P_{n,r} = \binom{n}{r}$.

Proof. By induction on n.

Base Case. For $n = 0$, the only case to consider is $r = 0$, for which we have $\binom{0}{0} = 1 = P_{0,0}$.

Inductive Step. Assume that for all $0 \leq r \leq n-1$ it holds $P_{n-1,r} = \binom{n-1}{r}$. We want to prove that for all $0 \leq r \leq n$ it holds $P_{n,r} = \binom{n}{r}$. We consider three cases.

1. If $r = 0$, then $P_{n,0} = 1 = \binom{n}{0}$.
2. If $r = n$, then $P_{n,n} = 1 = \binom{n}{n}$.
3. If $1 \leq r \leq n-1$, then we have

$$\begin{aligned}
P_{n,r} &= P_{n-1,r} + P_{n-1,r-1} && \text{by definition of } P_{n,r} \\
&= \binom{n-1}{r} + \binom{n-1}{r-1} && \text{using the inductive hypothesis} \\
&= \binom{n}{r} && \text{by Lemma 1.1}
\end{aligned}$$

∎

Recall that for $0 \le r \le n$,

$$\binom{n}{r} = \binom{n}{n-r}.$$

The next theorem gives the ratio between successive binomial coefficients (same n, successive r):

Theorem 1.3 *Suppose* $1 \le r \le n$. *Then*

$$\binom{n}{r} = \frac{(n-r+1)}{r}\binom{n}{r-1}.$$

Proof. [Combinatorial] Consider the number of ways of choosing a committee of r members out of n people, and also choose a chairman for the committee. There are $\binom{n}{r}$ ways of choosing the committee, and for each such committee there are r choices for the chair. Alternatively, there are $\binom{n}{r-1}$ ways of choosing all the members but the chair, and then $n-r+1$ ways to choose the chair among the remaining people. It follows that

$$r\binom{n}{r} = (n-r+1)\binom{n}{r-1}$$

■

Proof. [Algebraic]

$$\begin{aligned}
\binom{n}{r} \Big/ \binom{n}{r-1} &= \frac{n!}{r!(n-r)!} \times \frac{(r-1)!(n-(r-1))!}{n!} \\
&= (n-r+1)/r.
\end{aligned}$$

■

Note that the ratio between successive binomial coefficients is > 1, which means the successive coefficients are increasing, as long as $r < \frac{n+1}{2}$. Also, the ratio is < 1, which means the coefficient are decreasing, when $r > \frac{n+1}{2}$. This says that the coefficients increase, reach the maximum, then decrease, i.e., they are *unimodal.* Here is another equation about binomial coefficients.

Theorem 1.4 *Suppose* $0 \le r \le m, n$. *Then*

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

Proof. Suppose there are m red balls and n blue balls, all distinct. There are $\binom{m+n}{r}$ ways to choose r balls from the two sets combined. This is the LHS of the identity. Now decompose this collection of choices based on how many of the chosen balls are blue and how many are red. For any k, $0 \leq k \leq r$, there are $\binom{m}{r-k}$ ways to choose $r-k$ red balls, and $\binom{n}{k}$ ways to choose k blue balls. By the product rule, that makes $\binom{m}{r-k}\binom{n}{k}$ ways to choose r balls such that k of them are blue. Adding up these numbers for all k we have that this is $\sum_{k=0}^{r}\binom{m}{r-k}\binom{n}{k}$ which is the RHS. ∎

2 Applications of the Binomial Theorem

Recall that the Binomial Theorem states that

$$(x+y)^n = \sum_{r=0}^{n}\binom{n}{r}x^r y^{n-r}$$

The Binomial Theorem has a couple of immediate corollaries.

Corollary 2.1 $\sum_{r=0}^{n}\binom{n}{r} = 2^n$.

Proof. Set $x = y = 1$ in the Binomial Theorem.

$$2^n = (1+1)^n = \sum_{r=0}^{n}\binom{n}{r}1^r 1^{n-r} = \sum_{r=0}^{n}\binom{n}{r}.$$

∎

There is also a combinatorial proof: $\sum_{r=0}^{n}\binom{n}{r}$ is equal to the total number of subsets of a set of n elements, and we know that there are 2^n subsets.

Corollary 2.2 $\sum_{r=0}^{n}(-1)^r\binom{n}{r} = 0$.

Proof. Set $x = -1$ and $y = 1$ in the Binomial Theorem.

$$0 = ((-1)+1)^n = \sum_{r=0}^{n}\binom{n}{r}(-1)^r 1^{n-r} = \sum_{r=0}^{n}(-1)^r\binom{n}{r}.$$

∎

Corollary 2.2 can be used to give a combinatorial proof of the Inclusion-Exclusion formula.

Theorem 2.3 (Inclusion-Exclusion)

$$|A_1 \cup \ldots \cup A_n| = \sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{S \subseteq \{1,\ldots,n\}, |S|=k} |\bigcap_{i \in S} A_i| \right) .$$

Proof. [Combinatorial] For any element $x \in A_1 \cup \ldots \cup A_n$, we will compute the contribution of x to the RHS of the inclusion-exclusion formula.

Let m be the number of sets among $A_1, \ldots, A_n$ to whom x belongs. Then, since x belongs to m sets, x contributes m to the RHS when $k = 1$; also, since x belongs to $\binom{m}{2}$ intersections of pairs of sets, x will contribute $-\binom{m}{2}$ to the RHS when $k = 2$. In general, for $k \leq m$, x belongs to $\binom{m}{k}$ intersections of k sets. Furthermore, x will give no contribution when $k > m$, since x belongs to no intersection of $m+1$ sets or more. Overall the contribution given by x is

$$\sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} \tag{1}$$

Since we know that

$$\sum_{k=0}^{m} (-1)^{k+1} \binom{m}{k} = - \left(\sum_{k=0}^{m} (-1)^{k} \binom{m}{k} \right) = 0 , \tag{2}$$

we can subtract the LHS of (2) from (1) and we conclude that the contribution given by x is

$$-(-1)^{0+1} \binom{m}{0} = 1$$

So, each element of $A_1 \cup \cdots \cup A_n$ contributes 1 to the inclusion-exclusion formula, which means that the inclusion-exclusion formula correctly computes the cardinality of $A_1 \cup \cdots \cup A_n$. ■

Proof. [By Boring Induction] We have already seen that for any two sets A and B it holds

$$|A \cup B| = |A| + |B| - |A \cap B| .$$

We will use this fact to prove the correctness of the Inclusion-Exclusion formula by induction on n.

Base Case. For $n = 1$ the Inclusion-Exclusion formula is just $|A_1| = |A_1|$.

Inductive Step. Assume that

$$|A_1 \cup \ldots \cup A_{n-1}| = \sum_{k=1}^{n-1} (-1)^{k+1} \left(\sum_{S \subseteq \{1,\ldots,n-1\}, |S|=k} |\bigcap_{i \in S} A_i| \right) .$$

and consider an n-th set A_n. Using Inclusion-Exclusion for two sets, we have

$$\begin{aligned}
|A_1 \cup \cdots \cup A_n| &= |(A_1 \cup \ldots \cup A_{n-1}) \cup A_n| \\
&= |A_1 \cup \ldots \cup A_{n-1}| + |A_n| - |(A_1 \cup \ldots \cup A_{n-1}) \cap A_n| \\
&= |A_1 \cup \ldots \cup A_{n-1}| + |A_n| - |(A_1 \cap A_n) \cup \ldots \cup (A_{n-1} \cap A_n)|
\end{aligned}$$

Using the inductive hypothesis we can compute unions of $n - 1$ sets.

$$\begin{aligned}
|A_1 \cup \ldots \cup A_{n-1}| &= \sum_{k=1}^{n-1} (-1)^{k+1} \left(\sum_{S \subseteq \{1,\ldots,n-1\}, |S|=k} |\bigcap_{i \in S} A_i| \right) \\
&= \sum_{k=1}^{n-1} (-1)^{k+1} \left(\sum_{S \subseteq \{1,\ldots,n\}, |S|=k, n \notin S} |\bigcap_{i \in S} A_i| \right)
\end{aligned}$$

$$\begin{aligned}
|(A_1 \cap A_n) \cup \ldots \cup (A_{n-1} \cap A_n)| &= \sum_{k=1}^{n-1} (-1)^{k+1} \left(\sum_{S \subseteq \{1,\ldots,n-1\}, |S|=k} |\bigcap_{i \in S} (A_i \cap A_n)| \right) \\
&= \sum_{k=1}^{n-1} (-1)^{k+1} \left(\sum_{S \subseteq \{1,\ldots,n-1\}, |S|=k} |\bigcap_{i \in S \cup \{n\}} A_i| \right) \\
&= -\sum_{k=2}^{n} (-1)^{k+1} \left(\sum_{S \subseteq \{1,\ldots,n\}, |S|=k, n \in S} |\bigcap_{i \in S} A_i| \right)
\end{aligned}$$

Putting the pieces together, we have that

$$
\begin{aligned}
|A_1 \cup \cdots \cup A_n| &= \sum_{k=1}^{n-1} (-1)^{k+1} \left(\sum_{S \subseteq \{1,\ldots,n\}, |S|=k, n \notin S} |\bigcap_{i \in S} A_i| \right) \\
&\quad + |A_n| \\
&\quad + \sum_{k=2}^{n} (-1)^{k+1} \left(\sum_{S \subseteq \{1,\ldots,n\}, |S|=k, n \in S} |\bigcap_{i \in S} A_i| \right) \\
&= \sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{S \subseteq \{1,\ldots,n\}, |S|=k} |\bigcap_{i \in S} A_i| \right)
\end{aligned}
$$

■

3 Balls and Bins

3.1 Identical vs Distinguishable Balls and Bins

In lecture we saw two results on the number of ways of putting balls into bins under different assumptions.

1. There are n^r ways to put r *distinct* balls into n *distinct* bins.
2. There are

$$\binom{n+r-1}{r}$$

ways to put r balls into n bins when the balls are *identical* and the bins are *distinct.*

There are other two cases that could be of interest: when the balls are *distinct* and the bins are *identical*, and when both balls and bins are *identical.* In these two cases there is no simple formula, and in general the problem has to be solved by case analysis (see Figure 1.)

For example, consider the problem of putting 5 different balls into 3 identical bins (i.e. the problem of dividing 5 distinct object into at most 3 groups). The partition of balls into bins can have one of the following patterns:

	Bins are distinct	Bins are identical
Balls are distinct	n^r	case analysis
Balls are identical	$\binom{n+r-1}{r}$	case analysis

Figure 1: Number of ways of putting r balls into n bins.

- $(5, 0, 0)$: all the balls are in the same bin. This case arises only once.
- $(4, 1, 0)$: a bin is empty, a bin gets one ball, the other bin gets the remaining 4 balls. There are 5 ways to get this case (corresponding to the number of ways to choose the ball that goes alone in one bin).
- $(3, 2, 0)$: this case arises $\binom{5}{3} = 10$ times (corresponding to the number of ways to choose the three balls that go together in one bin).
- $(3, 1, 1)$: this case arises $\binom{5}{3} = 10$ times.
- $(2, 2, 1)$: this case arises $4 \cdot \binom{4}{2} \cdot \frac{1}{2} = 12$ times. This is the trickier case. There are 5 ways to choose the ball that goes alone in one bin, and then there are $\binom{4}{2} \cdot \frac{1}{2} = 3$ ways to partition the remaining 4 balls into two indistinguishable bins so that there are two balls in every bin. We have to divide by two since $\binom{4}{2}$ counts the number of subsets of size 2 of a set of 4 elements, which is also the number of ways to arrange 4 distinct balls into 2 *distinct* bins.

Thus, there are $1 + 5 + 10 + 10 + 15 = 41$ ways to put 5 different balls into 3 identical bins.

What is the number of ways to put 5 identical balls into 3 identical bins? We have to repeat the previous case analysis, however now each of the previous cases can occur only once, so the answer is simply 5.

3.2 At Least a Ball in Each Bin, at Most k Balls in Each Bin

In how many ways can we put r identical balls into n distinct bins so that there is at least a ball in any bin (assuming $r > n$)? This is equivalent to look for a positive integer solution of the equation $x_1 + x_2 + \ldots + x_r = n$.

Theorem 3.1 *Assume $r > n$; the number of ways of putting r identical balls into n distinct bins so that each bin gets at least one ball is*

$$\binom{r-1}{n-1}$$

Proof. [Combinatorial] Since the balls are identical, and we want at least one of them in every bin, we can start by putting one ball in every bin. Then, any way of putting the remaining $r-n$ balls into the n bins gives a legal solution. Therefore, the answer is given by the number of ways of putting $r-n$ identical balls into n distinct bins, that is

$$\binom{n+(r-n)-1}{r-n} = \binom{r-1}{r-n} = \binom{r-1}{n-1}$$

where the last equality is due to the fact that $\binom{n}{k} = \binom{n}{n-k}$. ■

If, instead of having a lower bound on the number of balls to put in any bin, we have an upper bound, then there is again no simple formula. Suppose that we want to count the ways of putting 5 identical balls into 3 distinct bins so that each bin gets at most 3 balls. It turns out that it is easier to count the number of illegal ways of putting the balls into the bins (i.e. the ways in which a bin gets 4 balls or more) and then subtract this number from the total.

The total number of ways of putting the 5 identical balls into 3 distinct bins is $\binom{5+3-1}{2} = \binom{7}{2} = 21$.

Illegal ways of putting the balls are to put all the 5 balls in one bin (this arises 3 times) and to put 4 balls in one bin and 1 ball in another (this arises 6 times, since there are 3 choices for the bin that receives 4 balls and 2 choices for the bin that receives 1 ball). Thus, there are 9 illegal cases, and therefore 12 legal cases.

Similarly, it is possible to deal with the case of putting r balls in n bins so that each bin receives at least one and no more than k balls. We first put one ball in each bin, and then we have to put $r-n$ balls into n bins in such a way that no bin gets more than $k-1$ balls. This reduces to the previous case.

4 Team Problem

In a company there are n engineers and m projects (and, thankfully, there are more engineers than projects.) Each engineer works on exactly one project, and each project must have at least one engineer assigned to it. In how many ways can the boss assign projects to the engineers?

The above problem is equivalent to the problem of counting the number of surjective functions $f : A \to B$ between two non-empty sets A and B such that $|A| = n$ (engineers) and $|B| = m$ (projects).

1. Find the number of functions $f : A \to B$.

2. Use Inclusion-Exclusion to derive a formula expressing the number of non-surjective functions $f : A \to B$ in terms of binomial coefficients.

3. Use the previous steps to prove that the number of surjections $f : A \to B$ is

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n$$

where $n = |A|$ and $m = |B|$.

Solution

Let $n = |A|$, $m = |B|$. We compute the total number of functions $f : A \to B$.

Lemma 4.1 *The total number of functions $f : A \to B$ is m^n.*

Proof. [Inductive] By induction on n.

Base Case. $n = 1$. Let $A = \{a\}$. Then a function $f : \{a\} \to B$ is totally specified by the value $f(a)$. There are m choices.

Inductive Step. Assume that for all A' such that $|A'| = n-1$ it is true that there are m^{n-1} functions $f' : A' \to B$. Fix an element $a \in A$ and consider the set $A' = A - \{a\}$. By the inductive hypothesis, there are m^{n-1} functions $f' : A' \to B$. For any such function, and for any $b \in B$, we can define a function $f : A \to B$ as follows

$$f(x) = \begin{cases} b & \text{If } x = a; \\ f'(x) & \text{otherwise.} \end{cases}$$

Indeed, there is a bijection between functions $f : A \to B$ and pairs (f', b) where $f' : A' \to B$ and $b \in B$. By the product rule, the number of such pairs is $m^{n-1} \cdot m = m^n$, therefore there are m^n functions $f : A \to B$. ∎

Proof.[Combinatorial] Functions $f : A \to B$ are in $1-1$ correspondence with sequences in $B^{|A|}$. By the product rule, the size of $B^{|A|}$ is m^n. ∎

Recall that for a function $f : A \to B$, the *range* of f is defined as

$$range(f) = \{b \in B : \exists a \in A. f(a) = b\} .$$

We say that f is not surjective when $range(f) \neq B$, meaning that there exists a $b \in B$ such that $b \notin range(f)$. Let F_b be the set of functions $f : A \to B - \{b\}$. Clearly the set of non-surjective functions is given by $\bigcup_{b \in B} F_b$. The number of non-surjective functions can now be computed using inclusion-exclusion

$$|\bigcup_{b\in B} F_b| = \sum_{k=1}^{|B|}(-1)^{k+1} \sum_{S\subseteq B, |S|=k} |\bigcap_{b\in S} F_b| \tag{3}$$

It remains to compute the cardinality of the intersections. Observe that $\bigcap_{b\in S} F_b$ is just the set of functions $f : A \to B - S$, and there are $(m - |S|)^n$ such functions. The number of non-surjective functions is thus

$$\begin{aligned}
\sum_{k=1}^{|B|}(-1)^{k+1} \sum_{S\subseteq B, |S|=k} |\bigcap_{b\in S} F_b| &= \sum_{k=1}^{m}(-1)^{k+1} \sum_{S\subseteq B, |S|=k} (m-k)^n \\
&= \sum_{k=1}^{m}(-1)^{k+1} \binom{m}{k} (m-k)^n \ .
\end{aligned}$$

The number of surjective functions is

$$m^n - \sum_{k=1}^{m}(-1)^{k+1} \binom{m}{k} (m-k)^n = m^n + \sum_{k=1}^{m}(-1)^{k} \binom{m}{k} (m-k)^n = \sum_{k=0}^{m}(-1)^{k} \binom{m}{k} (m-k)^n$$

In the special case where $|A| = |B| = n$, a function $f : A \to B$ is a surjection if and only if it is a bijection. The number of bijections between two sets of size n is $n!$. Therefore we have the remarkable equation

$$n! = \sum_{k=0}^{n}(-1)^{k} \binom{n}{k} (n-k)^n$$