Massachusetts Institute of Technology
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Professor Albert R. Meyer

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Notes for Recitation 5 (3/6/98)

1 Asymptotic Notations

There is need to compare growth rates of functions in many parts of Mathematics and Computer Science, and there are about a half-dozen standard ways to make such comparisons. For the time being, we'll stick to $O()$, $\Theta()$, and $\Omega()$ which were introduced in lecture:

Definition 1.1 *For functions $f, g : \mathbb{R} \to \mathbb{R}$, define*

$$f = O(g) \textit{ iff } \exists c > 0 \exists x_0 \forall x \geq x_0 : |f(x)| \leq cg(x).$$

Also

$$f = \Theta(g) \textit{ iff } [f = O(g) \textit{ and } g = O(f)],$$

and

$$f = \Omega(g) \textit{ iff } g = O(f).$$

The value of these notations is that they highlight growth rate and allow suppression of distracting factors and low-order terms. For example, if the running time of an algorithm is

$$T(n) = 10n^3 - 20n^2 + 1,$$

then

$$T(n) = \Theta(n^3).$$

$T(n)$ grows *cubically*: it is bounded both above and below by functions of the form cn^3.

Another such example is

$$\pi^2 3^{x-7} + \frac{(2.7x^{113} + x^9 - 86)^4}{\sqrt{x}} - 1.08^{3x} = \Theta(3^x).$$

Lemma 1.2 *If $f = O(g)$, then $af = O(bg)$ for any $a, b > 0$*

Exercise: Prove Lemma 1.2.

If $l(x)$ is any function which increases unboundedly — $\ln x$ is a good example — then $x/l(x) < x$ as soon as x gets large enough that $l(x) > 1$. In particular, $x/l(x) = O(x)$. Intuitively, $f = O(g)$ means that, to within a constant factor, g grows at least as fast as f. In this case, the g grows strictly faster:

Theorem 1.3

$$\textit{It is not true that } x = O(x/\ln x).$$

Proof. We must show that for all $c > 0$, there does not exist an x_0 such that for all $x \geq x_0$, we have $|x| \leq cx/\ln x$. But $x > cx/\ln x$ for all $x > e^c$, so there can't be such an x_0. ∎

Notice that, again, the only fact we used about $\ln x$ in the above proof is that it is an unbounded increasing function.

A more significant fact about the growth of $\ln x$ is

Theorem 1.4

$$\forall a > 0 : \ln x = O(x^a).$$

Proof. Remember L'Hôpital's rule: if $\mathrm{Lim}_{x\to\infty} f(x) = \infty = \mathrm{Lim}_{x\to\infty} g(x)$, then

$$\mathrm{Lim}_{x\to\infty} f(x)/g(x) = \mathrm{Lim}_{x\to\infty} f'(x)/g'(x)$$

where f', g' are the derivatives of f, g, and f, g, f', g' are continuous. So

$$\mathrm{Lim}_{x\to\infty} \frac{\ln x}{x^a} = \mathrm{Lim}_{x\to\infty} \frac{1/x}{ax^{a-1}} = \mathrm{Lim}_{x\to\infty} \frac{1}{ax^a} = 0.$$

The last equality follow from the familiar fact – which we'll take for granted – that that for all $a > 0$, the value of x^a grows unboundedly as x approaches infinity. Since the limit is 0, we can be sure that once x is large enough, $\ln x/x^a \leq 1$, that is $\ln x \leq x^a$. So indeed $\ln x = O(x^a)$. ∎

Similarly,

Theorem 1.5

$$\forall a > 1 : x = O(a^x).$$

Proof. We know that $\ln a > 0$, so in the Taylor series

$$a^x = a^0 + x(\ln a)a^0 + x^2(\ln a)^2 a^0/2 + \ldots$$

all the coefficients are positive. Hence

$$a^x > 1 + (\ln a)x > (\ln a)x \text{ for } x > 0.$$

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Asymptotics of Powers and Exponentials

We include here notes on a few more basic facts about growth of powers and exponentials which were not presented recitation.

Corollary 1.6

$$\forall k \in \mathbb{N}, a > 1 : y^k = O(a^y).$$

Proof. By induction on k. In the base case $k = 0$, we have $y^k = y^0 = 1 < a \le a^y$ for $y \ge 1$, so $y^0 = O(a^y)$. Notice that the only facts we have used here about a^y are that $a^1 = a$ and a^y is a nondecreasing function of y.

We now assume the induction hypothesis $y^k = O(b^y)$ for all $b > 1$ and prove $y^{k+1} = O(a^y)$ for all $a > 1$. For any $a > 1$, let $b = \sqrt{a}$, so also $b > 1$. By hypothesis, $y^k \le cb^y$ for some $c > 0$ and all $y \ge$ some y_0. Also, by Theorem 1.5, $y \le c'b^y$ for some $c' > 0$ and all $y \ge$ some y'_0. So

$$\begin{aligned} y^{k+1} &= y(y^k) \le c'b^y(cb^y) = cc'(b^y)^2 \\ &= cc'b^{2y} = cc'(b^2)^y = cc'a^y \end{aligned}$$

for $y \ge \max(y_0, y'_0)$, which shows that $y^{k+1} = O(a^y)$. ∎

Theorems 1.4 and 1.5 are important enough that it's worth examining another way to prove them. This time, we'll do it without appeal to calculus, starting with:

Lemma 1.7 *If* $Lim_{x\to\infty} g(x) = \infty$ *and* $f = O(g)$, *then* $\ln f = O(\ln g)$.

Proof. Since $f = O(g)$, there are $c > 0$ and x_0 such that $f(x) \le cg(x)$ for all $x \ge x_0$. Since $g(x)$ goes to ∞ with x, so does $\ln g(x)$ and in particular, $\ln g(x) \ge 1$ for all $x >$ some x_1. So for $x \ge \max(x_0, x_1)$,

$$\ln f(x) \le \ln cg(x) = \ln c + \ln g(x) \le (\ln c + 1)\ln g(x),$$

and hence $\ln f(x) = O(\ln g(x))$. ∎

We'll prove Theorem 1.4 by showing that it follows from Corollary 1.6 and Lemma 1.7. To see this, for $a > 0$, let $k = \lceil 1/a \rceil$. From Theorem 1.6, we have $y^k = O(e^y)$. If we change variables, letting $x = y^k$, then it follows from the definition of $O()$ that $x = O(e^{x^{1/k}})$. Since $1/k \le a$, also $x^{1/k} \le x^a$, so $x = O(e^{x^a})$. Now by Lemma 1.7, $\ln x = O(x^a)$.

Since Corollary 1.6 followed from Theorem 1.5, all that's left to show is a noncalculus proof of Theorem 1.5. We'll again use the fact that a^x is nondecreasing, which implies that if $n = \lfloor x \rfloor$, then $a^n \le a^x$. Since $x = O(\lfloor x \rfloor)$, Theorem 1.5 will follow if we prove

$$\forall a > 1 : n = O(a^n)$$

where n is a variable ranging over $\mathbb{N}$. But this follows immediately from:

Lemma 1.8 *For all $\epsilon > 0$ and $n \in \mathbb{N}$,*

$$n\epsilon \leq (1+\epsilon)^n.$$

Proof. We will prove by induction the stronger fact that

$$1 + n\epsilon \leq (1+\epsilon)^n.$$

The base case $n = 0$ is immediate.

So suppose $1 + n\epsilon \leq (1+\epsilon)^n$. Multiplying both sides of this inequality by $1+\epsilon$ yields

$$(1+\epsilon)(1+n\epsilon) \leq (1+\epsilon)^{n+1}$$

But

$$(1+\epsilon)(1+n\epsilon) = 1 + (n+1)\epsilon + n\epsilon^2 \geq 1 + (n+1)\epsilon,$$

so

$$1 + (n+1)\epsilon \leq (1+\epsilon)^{n+1}.$$

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A final note about asymptotic notation. Often one sees expressions like:

$$(1+1/x)^n = 1 + n/x + O(1/x^2).$$

The above expression really means that

$$(1+1/x)^n - 1 - n/x = O(1/x^2).$$

That is, the error term in the approximation for $(1+1/x)^n$ is at most a constant divided by x^2.

Other asymptotic relations

There are three other asymptotic notations in common use. We include them here for the record, but they will not be used further this term.

The tightest of the relationships between growth rates is *asymptotic equality*, $\sim$:

Definition 1.9

$$f \sim g \text{ iff } Lim_{x\to\infty}\frac{f(x)}{g(x)} = 1.$$

This was the relation described in lecture for the Harmonic number

$$H_n \sim \ln n.$$

For example, $x + 1/x \sim x - \sqrt{x}$. Note that if the values of f and g are nonnegative, then $f \sim g \implies f = \Theta(g)$.

Exercise: Give an example of f, g such that $f = \Theta(g)$ but **not** $f \sim g$.

To indicate that g actually grows even faster than f, we can use "Little Oh" notation:

Definition 1.10 $f = o(g)$ *iff* $\forall c > 0 \exists x_0 \forall x \geq x_0 : |f(x)| \leq cg(x)$.

This definition is a little hard to follow because of all the quantifiers. For functions, g, which are nonnegative — which is generally all we care about in this context — a clearer way to say $f = o(g)$ is:

$$\mathrm{Lim}_{x \to \infty} \frac{f(x)}{g(x)} = 0.$$

For example, the proof of Theorem 1.3 above actually yielded a the stronger result:

Theorem 1.11

$$\frac{x}{\ln x} = o(x).$$

Theorem 1.12

$$\textit{It is not true that } \frac{x}{100} = o(x).$$

Proof. The proof is by contradiction. Assume for the purpose of contradiction that $x/100 = o(x)$. Then for all $c > 0$, there exists an x_0 such that for all $x \geq x_0$, $|x/100| \leq cx$. In particular, if $c = 1/200$, there must exist an x_0 such that for all $x \geq x_0$, $|x/100| \leq x/200$. Since this statement is false for all positive x, we have reached a contradiction. ∎

Another, rarely used, asymptotic notation is "Little Omega", $\omega()$, which is just the opposite of "Little Oh", $o()$.

Definition 1.13 $g = \omega(f)$ *iff* $f = o(g)$

For example, $x^2 = \omega(x)$ since x^2 grows strictly faster than x.

2 Can sum 'em

2.1 Perturbation Method

We have seen some methods to find closed forms of sums. Finding closed form solutions is tricky business because it is not always the case that they exist! Let's carefully review the way to discover summation formulas called the *perturbation* method in lecture. As with all other methods it only works sometimes.

Consider the sum, $S_n = \sum_{i=0}^{n} ix^i$.

The general approach is to add the $(n+1)$st term to both sides,

$$\begin{aligned} S_n + (n+1)x^{n+1} &= 0x^0 + \textstyle\sum_{i=1}^{n+1} ix^i \\ &= \textstyle\sum_{i=0}^{n} (i+1)x^{i+1} \\ &= \textstyle\sum_{i=0}^{n} ix^{i+1} + \sum_{i=0}^{n} x^{i+1} \\ &= x\textstyle\sum_{i=0}^{n} ix^i + x\sum_{i=0}^{n} x^i \\ &= xS_n + x(1 - x^{n+1})/(1 - x) \end{aligned}$$

Having found S_n in the last expression, bring terms involving S_n to one side and solve,

$$\begin{aligned}(1-x)S_n &= \frac{x(1-x^{n+1}) - (n+1)x^{n+1}(1-x)}{(1-x)} \\ S_n &= \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}\end{aligned}$$

2.2 Integral Method

Let us now go back to the integral method and see how to evaluate the sum

$$\sum_{i=1}^{\infty} \frac{1}{i^2}$$

In a similar fashion to the example in the lecture, this sum, minus its first term, is less than the area under the curve $y = 1/x^2$ from 1 to ∞. That it,

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{1}{i^2} &= 1 + \sum_{i=2}^{\infty} \frac{1}{i^2} \\ &\leq 1 + \int_1^{\infty} x^{-2} dx \\ &= 2.\end{aligned}$$

Thus, our infinite sum is at most 2, and in particular it converges. In fact, one can prove (using complex analysis) that the exact value of the sum is $\pi^2/6$; we won't prove this here.

2.3 Formal Manipulation

Next let's look at the following *doubly infinite* sum

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x^i y^j,$$

where (critically)

$$0 < x, y < 1.$$

We can therefore re-write this as

$$\begin{aligned}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} x^i y^j &= \sum_{i=0}^{\infty} x^i \left(\sum_{j=0}^{\infty} y^j\right) \\ &= \sum_{i=0}^{\infty} x^i \left(\frac{1}{1-y}\right) \\ &= \left(\frac{1}{1-y}\right)\sum_{i=0}^{\infty} x^i \\ &= \frac{1}{(1-x)(1-y)}.\end{aligned}$$

3 Can't sum it!

We note that the manipulations above were justified because we had a series of positive real numbers whose sum converged. Just to reinforce this, here's a case in which re-arrangements of the sum are clearly not allowed. Let

$$S = \sum_{i=0}^{\infty} (-1)^i = 1 + (-1) + 1 + (-1) + \cdots .$$

Consider the following:

$$S = [1 + (-1)] + [1 + (-1)] + \cdots = 0 + 0 + \cdots = 0.$$

$$S = 1 + [(-1) + 1] + [(-1) + 1] + \cdots = 1 + 0 + 0 + \cdots = 1.$$

What's the problem here?

If we define $S_n = \sum_{i=0}^{n} (-1)^i$ then $S_n = 1$ for odd n and $S_n = 0$ for even n. Hence S_n does not have a well-defined limit as n tends to infinity. S is not a convergent sum.

4 Team Problems

Problem 1. We begin with two large glasses. The first glass contains a pint of water, and second contains a pint of wine. We pour 1/3 of a pint from the first glass into the second, stir up the wine/water mixture in the second glass, and then pour 1/3 of a pint of the mix back into the first glass. We now repeat this pouring back-and-forth process a total of n times.

(a) Describe a closed form formula for the amount of wine in the first glass after n back-and-forth pourings.

(b) What is the limit of the amount of wine in each glass as n approaches infinity?

Solution: The state of the system of glasses/wine/water at the beginning of a round of pouring and pouring back is determined by the total amount of wine in the first glass. Initially, the first glass contains one pint of wine and no water. Suppose the first glass contains w pints of wine, $0 \leq w \leq 1$. Since we're at the beginning of a round, it contains one pint of liquid, and therefore contains $1 - w$ pints of water. The second glass contains the rest of the wine and water.

Pouring 1/3 pint from first glass to second leaves 2/3 pint of liquid and $(2/3)w$ wine in the first glass, and 4/3 pints liquid and $1 - (2/3)w$ wine in the second glass. Pouring 1/3 pint back from second into first transfers a proportion of $(1/3)/(4/3)$ of the wine in the second glass into the first. So the round completes with both glasses containing a pint of liquid, and the first glass containing

$$(2/3)w + (1/4)(1 - (2/3)w) = 1/4 + w/2$$

pints of wine. After the second round, the first glass contains

$$1/4 + (1/4 + w/2)/2 = 1/4 + 1/8 + w/2^2$$

pints of wine, and after the nth round

$$\begin{aligned} w/2^n + \Sigma_{i=1}^n (1/2)^{i+1} &= w/2^n + (1/2)\Sigma_{i=1}^n (1/2)^i \\ &= w/2^n + (1/2)(-1 + \Sigma_{i=0}^n (1/2)^i) \\ &= w/2^n + (1/2)(-1 + (1 - (1/2)^{n+1})/(1 - 1/2)) \\ &= w/2^n - 1/2 + 1 - (1/2)^{n+1} \\ &= w/2^n + 1/2 - (1/2)^{n+1}. \end{aligned}$$

So when $w = 0$ initially, the pints of wine in the first glass is

$$1/2 - (1/2)^{n+1}.$$

You would expect after a thorough mixing that the glasses would contain the same amount of wine, viz. 1/2 pint. Indeed, the limiting amount of wine in the first glass approaches 1/2 from below as n approaches infinity. In fact, it approaches 1/2 no matter how the wine was initially distributed.

Problem 2. **(a)** Give an example of functions $f, g : \mathbb{N} \to \mathbb{N}$ such that neither $f = O(g)$ nor $g = O(f)$. Prove it.

Solution: $f(n) = 1 + n(n \bmod 2), g(n) = 1 + n((1 + n) \bmod 2)$. So $f(n)$ is n times the size of $g(n)$ on odd n, vice-versa on even n.

(b) Let $f_1, f_2, \ldots$ be a sequence of functions from $\mathbb{N}$ to $\mathbb{N}$ such that $f_i(n) = O(1)$ for all $i \geq 1$. Prove or give a counterexample:

$$\sum_{i=1}^{n} f_i(n) = O(2^n).$$

Solution: False, let f_i be a constant function with $f(n) = b(i)$ for all n, where $b(i)$ is nonnegative and not $O(2^i)$, for example, $b(i) = i2^i$. So $f_i = O(1)$ for each i. Now

$$\sum_{i=1}^{n} f_i(n) \geq f_n(n) = b(n) \neq O(2^n).$$